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CA-MATH-804: Numerical Analysis

Assignment Sheet 2. Due: March 27, 2022

Exercise 1 [5 Points]: Assuming $p, q = 1, 2, \infty, F$ recover the following table of equivalence constants c_{pq} such that $\forall A \in \mathbb{R}^{n\times n}$ we have $\|A\|_p \leq c_{pq} \|A\|_q$

c_{pq}	$q = 1$	$q = 2$	$q = \infty$	$q = F$
$p = 1$	1	$\sqrt{n}$	n	$\sqrt{n}$
$p = 2$	$\sqrt{n}$	1	$\sqrt{n}$	1
$p = \infty$	n	$\sqrt{n}$	1	$\sqrt{n}$
$p = F$	$\sqrt{n}$	$\sqrt{n}$	$\sqrt{n}$	1

Exercise 2 [5 Points]: For any square matrix $A \in \mathbb{R}^{n\times n}$, prove the following relations

a) $\frac{1}{n} K_2(A) \leq K_1(A) \leq n K_2(A)$,

b) $\frac{1}{n} K_\infty(A) \leq K_2(A) \leq n K_\infty(A)$,

c) $\frac{1}{n^2} K_1(A) \leq K_\infty(A) \leq n^2 K_1(A)$,

where $K_p(A) = \|A\|_p \|A^{-1}\|_p$. These relations show that if a matrix is ill-conditioned in a certain norm, it remains so even in another norm, up to a scaling factor.

Exercise 3 [5 Points]: Prove the following claims:

a) If $A \in \mathbb{R}^{n\times n}$ fulfils one of the following criteria:

1. strict row-sum criterion (strict diagonal dominance) $\sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| < |a_{jj}|$ for all j with $1 \leq j \leq n$.
2. strict column-sum criterion $\sum_{\substack{i=1\\ i\neq j}}^{n} |a_{ji}| < |a_{jj}|$ for all j with $1 \leq j \leq n$

then the Jacobi method converges for any initial guess $x^{(0)}$.

b) Let $A \in \mathbb{C}^{n\times n}$, then every eigenvalue λ of A fulfils one of the following inequalities

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1\\ j\neq i}}^{n} |a_{ij}| .$$

Exercise 4: A matrix in which the sum of the absolute values of the entries of a row is equal for every row, is called *row equilibrated*.

a) Show that every regular matrix A can be transformed into a row equilibrated matrix by multiplication with a regular diagonal matrix D.

b) Let A and D be as in a). Show that all non-singular diagonal matrices $\tilde{D}$ have

$$K_\infty (DA) \le K_\infty \left(\tilde{D}A\right).$$

Hint: Let $C = DA$ and find a lower estimate for the condition number of $\tilde{D}D^{-1}C$ in terms of the condition number of C.

Exercise 5: Prove Theorem 10 from class, which is repeated below.

Theorem. *For A, $\delta A \in \mathbb{R}^{n\times n}$ and b, $\delta b \in \mathbb{R}^n$, consider perturbations of the problem $Ax = b$. Assume there exists $\gamma > 0$ such that*

$$\|\delta A\| \le \gamma \|A\| \quad \text{and} \quad \|\delta b\| \le \gamma \|b\|$$

in suitable norms. Also let $\gamma K(A) < 1$ where $K(A)$ is the condition number of A in the norm used above. Then the perturbation δx of the solution fulfils

$$\frac{\|x\delta x\|}{\|x\|} \le \frac{1+\gamma K(A)}{1-\gamma K(A)} \quad \text{and} \quad \frac{\|\delta x\|}{\|x\|} \le \frac{2\gamma K(A)}{1-\gamma K(A)}.$$

Hints: Remember that $(A + \delta A)(x + \delta x) = b + \delta b$. Use

- compatibility (or consistency) of matrix/vector norms: $\|Ax\| \le \|A\| \|x\|$,
- sub-multiplicativity : $\|AB\| \le \|A\| \|B\|$

of the norms. Use a theorem from class which says something about the invertibilty of $\mathrm{Id} + B$ for matrices B,

Theorem. *Let $A \in \mathbb{R}^{n\times n}$, then*

$$\lim_{k\to\infty} A^k = 0 \iff \rho(A) < 1$$

Moreover, the geometric series $\sum_{k=0}^{\infty} A^k$ is convergent if and only if $\rho(A) < 1$. Then

$$\sum_{k=0}^{\infty} A^k = (I - A)^{-1}.$$

Then, if $\rho(A) < 1$, then $I - A$ is invertible and

$$\frac{1}{1+\|A\|} \le \left\|(I-A)^{-1}\right\| \le \frac{1}{1-\|A\|}$$

where $\|\cdot\|$ is an induced matrix norm such that $\|A\| < 1$.