CA-MATH-804: Numerical Analysis

Assignment Sheet 3. Due: April 18, 2022

Exercise 1 [5 Points]: Consider the matrix

$$A = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.$$

a) Show that for $\gamma \leq 0$ we have $K_\infty(A) = K_1(A) = (1+\gamma)^2$.

b) For the linear system $Ax = b$, where b is such that $x = (1-\gamma, 1)^T$ is the solution, find a bound for $\|\delta x\|_\infty / \|x\|_\infty$ in terms of $\|\delta b\|_\infty / \|b\|_\infty$ when $\delta b = (\delta_1, \delta_2)^T$. What can you say about the condition of the problem?

> **Definition.** *A **p-norm** for a vector is defined by* $\|x\|_p = \left(\sum_{i=0}^{n} |x_i|^p \right)^{1/p}$. *As* $p \to \infty$ *so the norm approaches the **infinity norm*** $\|x\|_\infty = \max_{0 \leq i \leq n} |x_i|$ *(also known as the maximum norm).*
> *For a matrix* $A \in \mathbb{R}^{n \times m}$, *then if a p-norm for vectors is used for both spaces* $x \in \mathbb{R}^n$ *and* $Ax \in \mathbb{R}^{n \times m}$, *then the corresponding **matrix norm** is*
> $$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}.$$
> *A **condition number** for a matrix can be defined for a given norm as* $K_p(A) = \|A\|_p \|A^{-1}\|_p$.

Exercise 2 [5 Points]: Check that the matrix $A = \mathrm{tridiag}_n(-1, \alpha, -1)$ with $\alpha \in \mathbb{R}$ has eigenvalues given by

$$\lambda_i = \alpha - 2\cos(i\theta), \quad i = 1, \ldots, n,$$

and the corresponding eigenvectors are

$$v_i = \left(\sin(i\theta), \sin(2i\theta), \ldots, \sin(ni\theta) \right)^T,$$

where $\theta = \dfrac{\pi}{n+1}$.

Exercise 3 [5 Points]: Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & 1 \end{pmatrix}$$

and $\alpha, \beta, \gamma \in \mathbb{R}$. Define an iterative method for $k \geq 0$ as

$$x^{(k+1)} = U^{-1}\left(b - Lx^{(k)} \right),$$

where

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}.$$

a) Find all values of α, β, γ such that the sequence of iterates $\{x^{(k)}\}$ converges for every initial guess $x^{(0)}$ and every right hand side.

b) Give an example for b leading to non-convergence in the case $\alpha = \beta = \gamma = -1$.

c) Is the solution always found with at most two iterations if $\alpha = \gamma = 0$?

Exercise 4 [3 × 2 Points]: Compute the following derivatives:

a) $\frac{\mathrm{d}}{\mathrm{d}s} \|x + sp\|_q^2$ for $x, p \in \mathbb{R}$ and $1 \leq q < \infty$,

b) $\nabla \|u(x)\|_2^2$ where $u : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth,

c) $\Delta \|x\|_2^2 := \operatorname{div}\left(\|x\|_2^2\right) := \nabla \cdot \left(\nabla \|x\|_2^2\right)$ for $x \in \mathbb{R}^n$.

Exercise 5 [5 Points]: Given $n+1$ distinct points $x_0, \dots, x_n \in \mathbb{R}$ consider the functions

$$l_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, \dots n.$$

Show that these functions form a basis for the polynomials $\mathbb{P}_n$ of degree less than or equal n over $\mathbb{R}$.