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CA-MATH-804: Numerical Analysis

Assignment Sheet 6. Due:

Exercise 1 [5 x 4* Points]: Consider the bending of a clamped beam subject to a transversal force f, which is described by the boundary value problem

$$\begin{aligned} u''''(x) &= f(x) \quad \text{in} \quad (0,1), \\ u(0) &= u(1) = 0, \\ u'(0) &= u'(1) = 0. \end{aligned}$$

a) Show that under certain conditions this problem is equivalent to the following variational (weak) problem:

$$(u'', v'') = (f, v) \quad \forall v \in W$$

where

$$W = \{v : (0,1) \to \mathbb{R} \,\|\, v \text{ and } v' \text{ are continuous}, v'' \text{ is piecewise continuous and } v(0) = v(1) = v'(0) = v'(1) = 0\}.$$

b) For an interval $I = [a,b]$ define

$$P_3(I) = \{v : I \to \mathbb{R} \,\|\, v \text{ is a polynomial of degree} \leq 3, \text{ i.e. } v(x) = a_3x^3 + a_2x^2 + a_1x + a_0 \text{ for } a_i \in \mathbb{R}\}.$$

Show that $v \in P_3(I)$ is uniquely defined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$ and determine the corresponding basis functions $b_i(x)$ such that

$$v(x) = v(a)b_0(x) + v'(a)b_1(x) + v(b)b_2(x) + v'(b)b_3(x).$$

c) Starting from b) use a uniform partitioning of $(0,1)$ to construct a finite dimensional subspace W_h of W consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions of W_h. What is the dimension of the resulting finite element space W_h?

d) Formulate a finite element method for the problem based on the space W_h. Find the corresponding system of equations.

e) Determine the finite element solution in the case of two intervals and $f = 1$. Compare with the exact solution.