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JTMS-MAT-13: Numerical Methods

Assignment Sheet 4. Released: April 24, 2024
Due: May 02, 2024

Exercise 1 [4+3+3 Points]:

(a) For computing $f'(x)$, consider the forward differencing scheme and apply Richardson extrapolation to derive an estimate with quadratic error term in the step-size h.

(b) Apply Richardson extrapolation to the estimate from **(a)** in order to derive an estimate with a better asymptotic error.

(c) Apply the estimates from **(a)** and **(b)** and the forward differencing scheme itself to compute $f'(\pi)$ for a function $f(x) = \cos(x)$ and step size $h = 0.3$.

Exercise 2 [3+3+3+3 Points]:

Consider the measurement values $p_0 = 7, p_1 = 6$, and $p_2 = 8$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$. Assume that a function $p(u) = \alpha \cos(u) + \beta u$ is supposed to approximate these values in the least squares sense.

(a) Show that the normal equations are given by

$$\alpha \sum_{i=0}^{2} \cos^2(u_i) + \beta \sum_{i=0}^{2} u_i \cos(u_i) = \sum_{i=0}^{2} p_i \cos(u_i),$$

$$\alpha \sum_{i=0}^{2} u_i \cos(u_i) + \beta \sum_{i=0}^{2} u_i^2 = \sum_{i=0}^{2} p_i u_i.$$

(b) Solve the normal equations for α and β.

(c) Compute the error in the L_2 sense that is minimized in **(b)**. What is the solution and what is the error if the last measurement value is now $p_2 = 5$?

(d) Do the same computations, but for the linear function $p_l(u) = \alpha u + \beta$ and compute the error in the L_2 sense. Is **(b)** or **(c)** the better approximation for this data set?

Exercise 3 [3 Points]:

Consider the function $f(x) = e^{x^2+1}$ and the set of equally spaced nodes $0, 0.25, 0.5, 0.75, 1$. The Newton form of a polynomial interpolant is given by

$$p_n(x) = \sum_{i=0}^{n} a_i n_i(x)$$

where the first polynomial is given by $n_0(x) = 1$ and

$$n_i(x) = (x - x_0)(x - x_1)\dots(x - x_{i-1}) \quad \text{for} \quad i > 0.$$

(a) Derive the polynomial $p_n(x)$ in Newton form that interpolates $f(x)$ at the given nodes.

Exercise 4 [2+3+2 Points]:

Consider the B-splines over the nodes $u_i \in \{0, 1, 2\}$, where $i = 0, 1, 2$.

(a) Sketch the B-splines $N_1^0(u)$ and $N_1^1(u)$.

(b) Use the recursive formulation of splines

$$N_i^0(u) = 1 \quad \text{if} \quad u \in [u_i, u_{i+1}) \quad \text{and} \quad 0 \quad \text{else,}$$

and

$$N_i^n(u) = \alpha_i^{n-1}(u) N_i^{n-1}(u) + \left(1 - \alpha_{i+1}^{n-1}(u)\right) N_{i+1}^{n-1}(u) \quad \text{with} \quad \alpha_i^{n-1}(u) = \frac{u - u_i}{u_{i+n} - u_i}$$

to compute the necessary splines for a linear spline interpolant with $n = 1$.

Compute the collocation matrix and solve the system of linear equations $s(u_i) = \sum_i c_i N_i^1(u_i)$ for the values $p_i \in \{5, -2, 2\}$. Where necessary, use for the external nodes $u_{-1} = -1$ and $u_3 = 3$ to construct the splines.

(c) Interpolate at $s(1.5)$.