Constructor University **Spring Semester 2024**

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JTMS-MAT-13: Numerical Methods

Assignment Sheet 6. Released: May 02, 2024
Due: May 17, 2024

Exercise 1 [4+2+2+3 Points]:

(a) For Gaussian quadrature on $[-1, 1]$ and $n = 3$, show that the four Gauss nodes $x_0, \dots, x_3$ as the roots of the polynomial $q(x)$ that is orthogonal to $1, x, x^2$ and x^3 are given by

$$x = \pm\sqrt{\frac{15 \pm 2\sqrt{30}}{35}}$$

(b) To illustrate the full quadrature, determine the first corresponding weight A_0 of the 4 point Gaussian quadrature.

(c) Based on your results of **(a)**, find the corresponding Legendre polynomial that uses the additional condition of $q(1) = 1$.

(d) Consider Gauss quadrature with 2 nodes on the interval $[-1, 1]$. Derive the resulting Legendre polynomial $q(x)$ as a solution to the orthogonality condition:

$$\int_{-1}^{1} q(x) x^k \mathrm{d}x = 0$$

for $k = 0, 1$. Determine the roots of $q(x)$ to arrive at the Gauss nodes. Use your result to approximate the integral:

$$\int_{-\frac{3\pi}{\sqrt{3}}}^{\frac{3\pi}{\sqrt{3}}} \sin(x) \cos^2(x) \mathrm{d}x$$

Remember that you need to convert into the appropriate interval. You may give the result either as an exact value or use a calculator to arrive at the final number. What kind of method is this?

Exercise 2 [3+4+4+2+2 Points]:

Consider the linear ordinary differential equation

$$y''(t) = -3y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

(a) Convert this 2^{nd} order ordinary differential equation into a system of two coupled first order ODEs, one in $y(t)$ and one in $y'(t)$. Write the system as a vector-valued ordinary differential equation in $\vec{v}(t) = (y(t), y'(t))^T$, in the form $f(\vec{v}) = A\vec{v}$.

(b) Show that the backward Euler method can be written as

$$\vec{u}_{n+1} = (I - hA)^{-1} \vec{u}_n$$

and provide the full system for $\vec{u}_{n+1}$ for the ODE presented above.

(c) Noting that $f_n = A\vec{u}_n$, so that $f(\vec{v}_n + hf_n) = A\vec{v}_n + hA^2\vec{v}_n$, show that Heun's method for the case of the vector-valued ODE presented above is

$$\vec{v}_{n+1}(t) = \begin{pmatrix} 1 + \dfrac{h^2}{2} & \dfrac{h}{2}(1 - 3h) \\ \dfrac{h}{2}(1 - 3h) & 1 + \dfrac{h}{2}(10h - 3) \end{pmatrix} \vec{v}_n(t).$$

(d) Calculate $y(0.3)$ and $y'(0.3)$ using backward Euler method with $h = 0.15$.

(e) Calculate $y(0.3)$ and $y'(0.3)$ using Heun's method with $h = 0.15$.

Exercise 3 [5 Points]:

Consider the ordinary differential equation

$$y'(t) = -y(t) + \cos(t) \quad \text{with} \quad y(0) = 1.$$

(a) Showing all working, calculate four time steps of the approximate solution using RK4 with step size $h = 0.1$.