Dr D. Sinden

Calculus and Linear Algebra for Graduate Students MDE-MET-01

Assignment Sheet 4. Released: November 4, 2024
Due: November 14, 2024

1. **[5 points]** An $n \times n$ matrix is invertible, if and only if it has rank n. Explain why this is true.
2. **[5 points]** If an $m \times n$ matrix has rank r, then it has an $r \times r$ submatrix S that is invertible. Remove $m - r$ rows and $n - r$ columns to find an invertible submatrix inside A, B and C.
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 4 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
3. **[5+5 points]** Find the inverses of the given matrices, if they exist.
$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & 2 & 3 \end{pmatrix}.$$
4. **[5 points]** Find the orthogonal projection of the vector $\mathbf{b}$ onto the line through $\mathbf{a}$ where
$$\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$
5. **[5 points]** Project $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ onto the lines through $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Draw the projections $\mathbf{p}_1$ and $\mathbf{p}_2$ and add $\mathbf{p}_1 + \mathbf{p}_2$. Observe that the projections do not add to $\mathbf{b}$, because $\mathbf{a}_1$ and $\mathbf{a}_2$ are not orthogonal.
6. **[5 points]** Find the vector $\mathbf{x}$ that is the reflection of the vector $\mathbf{y}$ about the line through $\mathbf{a}$.
$$\mathbf{y} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$
7. **[5 points]** Check that the vectors
$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} -1 \\ -2 \\ -5 \end{pmatrix}$$
are orthogonal and then find the projection of the vector $\mathbf{x} = (2, 2, 2)^T$ onto the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$. Is it important for your method that $\mathbf{a}_1$ and $\mathbf{a}_2$ are orthogonal?
8. **[5 + 5 points]**
   (a) Gram-Schmidt: Find orthonormal vectors $\mathbf{q}_1$ and $\mathbf{q}_2$ in the plane spanned by
$$\mathbf{a} = \begin{pmatrix} 1 \\ 3 \\ 4 \\ 5 \\ 7 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} -6 \\ 6 \\ 8 \\ 0 \\ 8 \end{pmatrix}.$$
   (b) Which vector in this plane is closest to $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$?