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Calculus and Linear Algebra for Graduate Students MDE-MET-01

Assignment Sheet 5. Released: November 14 2024
Due: November 24 2024

1. **[5 + 5 points]**
   (a) If a 4×4 matrix has $\det(A) = \frac{1}{2}$, find $\det(2A)$ and $\det(-A)$.
   (b) If a 3×3 matrix has $\det(A) = -1$, find $\det\left(\frac{1}{2}A\right)$ and $\det(-A)$.

2. **[5 + 5 points]** For any positive integer n, let J_n be the $n \times n$ matrix with ones on the complementary diagonal and zeros everywhere else, i.e.,

$$J_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad J_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

   (a) Which row exchanges show that $\det J_3 = -1$ and $\det J_4 = 1$?
   (b) For $n = 5, 6, 7$, count the row exchanges to permute J_n to the identity matrix I_n. Generalize your approach for every size n. Compute $\det J_{101}$.

3. **[5 + 5 + 5 points]** Compute

$$\text{(a) } \det \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \qquad \text{(b) } \det \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 6 & 6 & 1 \\ -1 & 0 & 0 & 3 \\ 0 & 2 & 0 & 7 \end{bmatrix}, \qquad \text{(c) } \det \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix}.$$

4. **[5 + 5 points]** The $n \times n$ determinant C_n has ones above and below the main diagonal, i.e.

$$C_1 = |0|, \qquad C_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}, \qquad C_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{vmatrix}, \qquad C_4 = \begin{vmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}.$$

   (a) Compute the determinants C_1, C_2, C_3 and C_4.
   (b) By using cofactor expansion, find the relation between C_n and C_{n-1} and C_{n-2}, find C_{10}.

5. **[5 points]** Consider a system

$$\begin{cases} 2x + 6y + 2z = 0 \\ x + 4y + 2z = 0 \\ 5x + 9y = 1. \end{cases}$$

   Use Cramer's rule to solve it for y only.