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# CTMS-MAT-13: Numerical Methods

**Assignment Sheet 5. Released: 23 April 2025**
**Due: 5 May 2025**

**Exercise 1 [2+3+3+2 Points]**:

Given the following system of linear equations:

$$\begin{aligned} 10x_1 - 2x_2 + x_3 &= -1 \\ -6x_1 + 9x_2 + x_3 &= 2 \\ 4x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

(**a**) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next two iterates $\vec{x}_1$ and $\vec{x}_2$ of the Jacobi method.

(**b**) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next two iterates $\vec{x}_1$ and $\vec{x}_2$ of the Gauss-Seidel method.

(**c**) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next two iterates $\vec{x}_1$ and $\vec{x}_2$ of the method of successive over-relaxation with $\omega = 1.25$.

(**d**) Are the Jacobi and the Gauss-Seidel methods guarenteed to converge?

**Exercise 2 [2+3+2+3 Points]**:

Let $A = L + D + U$ be a non-singular matrix with $L$ being a lower triangular, $D$ being a diagonal, and $U$ being an upper triangular matrix.

(**a**) Show that for a linear system of equations $A\vec{x} = \vec{b}$ with solution $\vec{x}$, an iterative solver at iteration step $k$ with an approximation $\vec{x}_k$ has an error $\vec{e}_k = \vec{x} - \vec{x}_k$ which can be written as

$$\vec{e}_k = \left(I - Q^{-1}A\right)^k \vec{e}_0$$

with initial error $\vec{e}_0 = \vec{x} - \vec{x}_0$ for the initial guess $\vec{x}_0$.

(**b**) Show that the Jacobi iteration for solving $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}_{k+1} = D^{-1}\left(\vec{b} - (L + U)\vec{x}_k\right)$$

(**c**) Show that the Gauss-Seidel iteration for solving $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}_{k+1} = (D + L)^{-1}\left(\vec{b} - U\vec{x}_k\right)$$

(**d**) Derive how the successive over-relaxation iteration for solving $A\vec{x} = \vec{b}$ with weight $\omega$ can be rewritten in a similar form to the one in (**c**).

**Exercise 3 [2+2+3 Points]**:

For the function $f(x) = (x+1)\sin\left(\frac{\pi}{2}(x+1)\right)$, evaluate the integral

$$\int_0^1 f(x)\,\mathrm{d}x$$

using four subintervals for

**(a)** Trapezium rule,

**(b)** Simpson's rule and

**(c)** Compute the error in each case against the exact solution.

**Exercise 4 [4+4 Points]**:

**(a)** Show, by applying the Romberg algorithm to the trapezium method, that the next improvement is Simpson's Rule.

**(b)** For

$$\int_1^2 \frac{1}{x^3}\mathrm{d}x = \frac{3}{8}.$$

Compute for four intervals, the trapezium, Simpson and the next refinement of the Romberg algorithm.