

## CTMS-MAT-13: Numerical Methods

Assignment Sheet 6. Released: 30 April 2025

Due: 20 May 2025

**Exercise 1 [4+2+2+3 Points]:**

- (a) For Gaussian quadrature on  $[-1, 1]$  and  $n = 3$ , show that the four Gauss nodes  $x_0, \dots, x_3$  as the roots of the polynomial  $q(x)$  that is orthogonal to  $1, x, x^2$  and  $x^3$  are given by

$$x = \pm \sqrt{\frac{15 \pm 2\sqrt{30}}{35}}.$$

- (b) To illustrate the full quadrature, determine the first corresponding weight  $A_0$  of the 4 point Gaussian quadrature.
- (c) Based on your results of (a), find the corresponding Legendre polynomial that uses the additional condition of  $q(1) = 1$ .
- (d) Consider Gauss quadrature with 2 nodes on the interval  $[-1, 1]$ . Derive the resulting Legendre polynomial  $q(x)$  as a solution to the orthogonality condition:

$$\int_{-1}^1 q(x) x^k dx = 0$$

for  $k = 0, 1$ . Determine the roots of  $q(x)$  to arrive at the Gauss nodes. Use your result to approximate the integral:

$$\int_{-\frac{3\pi}{\sqrt{3}}}^{\frac{3\pi}{\sqrt{3}}} \sin^2(x) \cos(x) dx.$$

Remember to convert into the appropriate interval. You may give the result either as an exact value or use a calculator to arrive at the final number.

**Exercise 2 [3+4+4+2+2 Points]:**

Consider the linear ordinary differential equation

$$y''(t) = -\alpha y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

- (a) Convert this second-order ordinary differential equation into a system of two coupled first-order ODEs, one in  $y(t)$  and one in  $y'(t)$ . Write the system as a vector-valued ordinary differential equation for  $\vec{v}(t) = (y(t), y'(t))^T$ , in the form  $f(\vec{v}) = A\vec{v}$ .
- (b) Show that the backward Euler method can be written as

$$\vec{u}_{n+1} = (I - hA)^{-1} \vec{u}_n$$

and provide the full system for  $\vec{u}_{n+1}$  for the ODE presented above.

- (c) Noting that  $f_n = A\vec{u}_n$ , so that  $f(\vec{v}_n + hf_n) = A\vec{v}_n + hA^2\vec{v}_n$ , show that Heun's method for the case of the vector-valued ODE given above is

$$\vec{v}_{n+1}(t) = \begin{pmatrix} 1 + \frac{h^2}{2} & h\left(1 - \alpha\frac{h}{2}\right) \\ h\left(1 - \alpha\frac{h}{2}\right) & 1 - h\left(\alpha - \frac{h}{2}(1 + \alpha^2)\right) \end{pmatrix} \vec{v}_n(t).$$

(d) With  $\alpha = 3$ , calculate  $y(0.2)$  and  $y'(0.2)$  using the backward Euler method with  $h = 0.1$ .

(e) With  $\alpha = 3$ , calculate  $y(0.2)$  and  $y'(0.2)$  using Heun's method with  $h = 0.1$ .

**Exercise 3 [7 Points]:**

Consider the ordinary differential equation

$$y'(t) = -y(t) + \ln(t+1) \quad \text{with} \quad y(0) = 1.$$

Showing all working, calculate four time steps of the approximate solution using the fourth-order Runge-Kutta method with step size  $h = 0.1$ .