Dr D. Sinden

CTMS-MAT-13: Numerical Methods

Problem Sheet 4 Solutions. Released: April 10 2026

Exercise 1: Consider the function $f(x) = e^{x+1}$ and the set of equally spaced nodes $0, 1/3, 2/31$. The Newton form of a polynomial interpolant is given by

$$p_n(x) = \sum_{i=0}^{n} a_i n_i(x)$$

where the first polynomial is given by $n_0(x) = 1$ and

$$n_i(x) = (x - x_0)(x - x_1) \ldots (x - x_{i-1}) \quad \text{for} \quad i > 0.$$

Derive the polynomial $p_n(x)$ in Newton form that interpolates $f(x)$ at the four given nodes.

Exercise 2: Derive the four Lagrange polynomials for $f(x) = e^{x+2}$ evaluated at the set of equally spaced nodes $0, 1/3, 2/3, 1$.

Exercise 3:

a) At $t = (1, 2, 3)$ values are measured as $p = (2, 4, 4.5)$. Using Aitken's method, verify that when $t = 1.5$, the interpolated value is approximately $p = 3.875$

$$\begin{array}{lll} p_{0,0} = 2 & & \\ & p_{0,1} = 3 & \\ p_{1,0} = 4 & & p_{0,2} = 3.875 \\ & p_{1,1} = 3.75 & \\ p_{2,0} = 4.5 & & \end{array}$$

b) For data measured at $t = (1, 2, 3, 5)$, whose values are measured as $p = (2, 4, 4.5, 5)$, use Aitken's method to find the approximate value at $t = 1.5$.

Exercise 4: Consider the measurement values $p_0 = 5, p_1 = 4$, and $p_2 = 6$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$. Let the function $p(u) = \alpha \cos(u) + \beta u$ approximate the data in the least squares sense.

a) Show that the normal equations are given by

$$\alpha \sum_{i=0}^{2} \cos^2(u_i) + \beta \sum_{i=0}^{2} u_i \cos(u_i) = \sum_{i=0}^{2} p_i \cos(u_i),$$
$$\alpha \sum_{i=0}^{2} u_i \cos(u_i) + \beta \sum_{i=0}^{2} u_i^2 = \sum_{i=0}^{2} p_i u_i.$$

b) Solve the normal equations for α and β.

c) Compute the error in the L_2 sense that is minimized in **(b)**.

d) What is the solution and what is the error if the last measurement value is now $p_2 = 5$?