

CTMS-MAT-13: Numerical Methods

Problem Sheet 5. Released: 22 April 2026

Exercise 1:

- a) For Gaussian quadrature on $[-1, 1]$ and $n = 3$, show that the four Gauss nodes x_0, \dots, x_3 as the roots of the polynomial $q(x)$ that is orthogonal to $1, x, x^2$ and x^3 are given by

$$x = \pm \sqrt{\frac{15 \pm 2\sqrt{30}}{35}}.$$

- b) To illustrate the full quadrature, determine the first corresponding weight A_0 of the 4 point Gaussian quadrature.
- c) Based on your results of (a), find the corresponding Legendre polynomial that uses the additional condition of $q(1) = 1$.
- d) Consider Gauss quadrature with 2 nodes on the interval $[-1, 1]$. Derive the resulting Legendre polynomial $q(x)$ as a solution to the orthogonality condition:

$$\int_{-1}^1 q(x) x^k dx = 0$$

for $k = 0, 1$. Determine the roots of $q(x)$ to arrive at the Gauss nodes. Use your result to approximate the integral:

$$\int_{-\frac{3\pi}{\sqrt{3}}}^{\frac{3\pi}{\sqrt{3}}} \sin^2(x) \cos(x) dx.$$

Remember to convert into the appropriate interval. You may give the result either as an exact value or use a calculator to arrive at the final number.

- Exercise 2:** For the function $f(x) = (x + 1) \sin\left(\frac{\pi}{2}(x + 1)\right)$, evaluate the integral

$$\int_0^1 f(x) dx$$

using four subintervals for

- a) Trapezium rule,
- b) Simpson's rule and
- c) Compute the error in each case against the exact solution.

Exercise 3:

- a) Show, by applying the Romberg algorithm to the Trapezium method, that the next improvement is Simpson's Rule.
- b) For

$$\int_1^2 \frac{1}{x^3} dx = \frac{3}{8}.$$

Compute for four intervals, the trapezium, Simpson and the next refinement of the Romberg algorithm.