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CTMS-MAT-13: Numerical Methods

Problem Sheet 6. Released: 5 May 2026

Exercise 1: Consider the linear ordinary differential equation

$$y''(t) = -\alpha y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

a) Convert this second-order ordinary differential equation into a system of two coupled first-order ODEs, one in $y(t)$ and one in $y'(t)$. Write the system as a vector-valued ordinary differential equation for $\vec{v}(t) = \left(y(t), y'(t)\right)^T$, in the form $f(\vec{v}) = A\vec{v}$.

b) Show that the backward Euler method can be written as

$$\vec{u}_{n+1} = \left(I - hA\right)^{-1} \vec{u}_n$$

and provide the full system for $\vec{u}_{n+1}$ for the ODE presented above.

c) Noting that $f_n = A\vec{u}_n$, so that $f(\vec{v}_n + hf_n) = A\vec{v}_n + hA^2\vec{v}_n$, show that Heun's method for the case of the vector-valued ODE given above is

$$\vec{v}_{n+1}(t) = \begin{pmatrix} 1 + \dfrac{h^2}{2} & h\left(1 - \alpha\dfrac{h}{2}\right) \\ h\left(1 - \alpha\dfrac{h}{2}\right) & 1 - h\left(\alpha - \dfrac{h}{2}\left(1 + \alpha^2\right)\right) \end{pmatrix} \vec{v}_n(t).$$

d) With $\alpha = 2.5$, calculate $y(0.3)$ and $y'(0.3)$ using the backward Euler method with $h = 0.1$.

e) With $\alpha = 2.5$, calculate $y(0.3)$ and $y'(0.3)$ using the Crank-Nicolson method with $h = 0.1$.

Exercise 2: Consider the ordinary differential equation

$$y'(t) = -y(t) + \ln(t+1) \quad \text{with} \quad y(0) = 1.$$

Showing all working, calculate four time steps of the approximate solution using the standard fourth-order Runge-Kutta method with step size $h = 0.1$.

Exercise 3: With $h = \frac{1}{4}$ compute $y(1)$ using Heun's method for the ordinary differential equation given by

$$ty'(t) = y(t)\sin(t) \quad \text{with} \quad y(0) = 2.$$