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CTMS-MAT-13: Numerical Methods

Quiz 1. 25 March 2025

Answer *four* questions only. All questions carry equal marks. Please write you name at the top and please clearly indicate which questions are to be marked.

All trigonometric values are in radians.

Exercise 1: Show that the Taylor series, with remainder, for $\ln(1+x)$ about $x=0$ can be written as

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k + \frac{(-1)^n}{n+1} \frac{x^{n+1}}{(1+\xi_x)^{n+1}}.$$

The function is given by

$$f(x) = \ln(1+x), \qquad x > -1.$$

Its derivatives are

$$f^{(k)}(x) = \frac{(-1)^{k-1}(k-1)!}{(1+x)^k} \quad (k \geq 1), \quad \text{so} \quad f^{(k)}(0) = (-1)^{k-1}(k-1)!$$

By Taylor's theorem about $x=0$ with Lagrange remainder, for some ξ_x between 0 and x,

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(n+1)}(\xi_x)}{(n+1)!} x^{n+1}.$$

Since $f(0)=0$, the $k=0$ term vanishes, and we obtain

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} x^k + \frac{(-1)^n n!}{(n+1)!(1+\xi_x)^{n+1}} x^{n+1}.$$

Simplifying factorial terms gives

$$\ln(1+x) = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k} x^k + \frac{(-1)^n}{n+1} \frac{x^{n+1}}{(1+\xi_x)^{n+1}}.$$

Exercise 2: Express the number 20.3125

a) in binary, i.e. base 2

b) in base 4

c) in base 16 (using hexadecimal format: 0,1,2,3,4,5,6,7,8,9,a,b,c,d,e,f)

a) Binary (base 2):

$$20 = 2^4 + 2^2 \Rightarrow 10100_2, \qquad \frac{5}{16} = \frac{1}{4} + \frac{1}{16} \Rightarrow 0.0101_2,$$

$$20.3125_{10} = \boxed{10100.0101_2}.$$

b) Base 4: group binary by 2 bits, and convert each pair into base 4, appending a zero to the first term

$$10100.0101_2 = (01\,01\,00).(01\,01)_2 \Rightarrow 110.11_4,$$

$$20.3125_{10} = 1 \cdot 4^2 + 1 \cdot 4^1 + 0 \cdot 4^0 + 1 \cdot \frac{1}{4^1} + 1 \cdot \frac{1}{4^2} + = \boxed{110.11_4}.$$

c) Base 16 (group binary by 4 bits):

$$10100.0101_2 = (0001\,0100).(0101)_2 \Rightarrow 14.5_{16},$$

$$20.3125_{10} = \boxed{14.5_{16}}.$$

This is $1 \cdot 16^1 + 4 \cdot 16^0 + 5 \cdot 16^{-1}$

Exercise 3: Using Gaussian elimination, or otherwise, what is the solution to the linear system given by $A\boldsymbol{x} = \boldsymbol{b}$,

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 2 & -3 \\ 0 & 4 & 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}.$$

- [x] $\left(\frac{1}{6}, \frac{25}{12}, \frac{1}{6}\right)^T$
- [ ] $\left(\frac{1}{6}, 1, \frac{1}{6}\right)^T$
- [ ] $(1, 1, 1)^T$
- [ ] $\left(1, \frac{25}{12}, 1\right)^T$
- [ ] $\left(\frac{25}{12}, 1, \frac{25}{12}\right)^T$
- [ ] $\left(\frac{25}{12}, \frac{1}{6}, \frac{25}{12}\right)^T$

First perform the elementary row-operations to get the row-echelon form for the augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 2 & 2 & -3 & 4 \\ 0 & 4 & 4 & 9 \end{array}\right] \xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 2 & -13 & 2 \\ 0 & 4 & 4 & 9 \end{array}\right] \xrightarrow{R_3 \leftarrow R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 1 \\ 0 & 2 & -13 & 2 \\ 0 & 0 & 30 & 5 \end{array}\right].$$

Back-substitution gives

$$z = \frac{5}{30} = \frac{1}{6}, \qquad 2y - 13z = 2 \;\Rightarrow\; y = \frac{25}{12}, \qquad x + 5z = 1 \;\Rightarrow\; x = \frac{1}{6}.$$

$$(x, y, z) = \left(\frac{1}{6}, \frac{25}{12}, \frac{1}{6}\right).$$

Exercise 4: For the matrix A given by

$$A = \begin{pmatrix} 4 & -1 & 1 \\ -1 & 17/4 & 11/4 \\ 1 & 11/4 & 7/2 \end{pmatrix},$$

show that Cholesky matrix L such that $A = LL^T$, is given by

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1/2 & 2 & 0 \\ 1/2 & 3/2 & 1 \end{pmatrix}.$$

Let L be lower triangular

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

with $A = LL^T$. Now equate the terms of the A with the product LL^T. Firstly, $l_{11}^2 = a_{11}$,

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2$$

$a_{21} = l_{11}l_{21}$, thus

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{-1}{2} = -\frac{1}{2}$$

$a_{31} = l_{11}l_{31}$, so

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{1}{2}.$$

Similarly

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{\tfrac{17}{4} - \tfrac{1}{4}} = \sqrt{4} = 2$$
$$l_{32} = \frac{a_{32} - l_{31}l_{21}}{l_{22}} = \frac{\frac{11}{4} - \frac{1}{2}\cdot(-\frac{1}{2})}{2} = \frac{\frac{11}{4} + \frac{1}{4}}{2} = \frac{3}{2}.$$

Finally

$$\begin{aligned} l_{33} &= \sqrt{a_{33} - l_{31}^2 - l_{32}^2} \\ &= \sqrt{\tfrac{7}{2} - \tfrac{1}{4} - \tfrac{9}{4}} \\ &= \sqrt{1} = 1. \end{aligned}$$

Thus

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -\frac{1}{2} & 2 & 0 \\ \frac{1}{2} & \frac{3}{2} & 1 \end{pmatrix}.$$

An alternative answer is to simply show that

$$LL^T = \begin{pmatrix} 4 & -1 & 1 \\ -1 & \frac{17}{4} & \frac{11}{4} \\ 1 & \frac{11}{4} & \frac{7}{2} \end{pmatrix} = A.$$

Exercise 5: Consider the linear system $A\boldsymbol{x} = \boldsymbol{b}$, with

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix}.$$

What is the second iterate of the Jacobi scheme, $\boldsymbol{x}^{(k+1)} = \left(I - D^{-1}A\right)\boldsymbol{x}^{(k)} + D^{-1}\boldsymbol{b}$, where D is the diagonal matrix of A and $\boldsymbol{x}^{(0)} = (0,0,0)^T$?

- ○ $\left(0, \frac{1}{4}, -\frac{3}{4}\right)^T$
- ☒ $\left(\frac{1}{7}, -\frac{5}{14}, \frac{3}{7}\right)^T$
- ○ $(1,1,1)^T$
- ○ $\left(-\frac{1}{3}, 0, -\frac{4}{7}\right)^T$
- ○ $\left(\frac{1}{3}, 0, \frac{4}{7}\right)^T$
- ○ $\left(\frac{1}{4}, -\frac{5}{8}, \frac{3}{4}\right)^T$

The diagonal matrix D is given by

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{pmatrix} \Rightarrow D^{-1} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}.$$

Thus

$$D^{-1}A = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ 3 & 6 & 2 \\ 3 & 3 & 7 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{3} \\ \frac{3}{7} & \frac{3}{7} & 1 \end{pmatrix}$$

so that

$$I - D^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -\frac{1}{3} & \frac{1}{3} \\ \frac{1}{2} & 1 & \frac{1}{3} \\ \frac{3}{7} & \frac{3}{7} & 1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{3} \\ -\frac{3}{7} & -\frac{3}{7} & 0 \end{pmatrix}$$

and

$$D^{-1}\boldsymbol{b} = \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{7} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix}.$$

Thus, the first approximation is

$$\boldsymbol{x}_1 = \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{3} \\ -\frac{3}{7} & -\frac{3}{7} & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix}.$$

Then, the second iterate is given by

$$\begin{aligned} \boldsymbol{x}_2 &= \begin{pmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{2} & 0 & -\frac{1}{3} \\ -\frac{3}{7} & -\frac{3}{7} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{4}{21} \\ -\frac{5}{14} \\ -\frac{1}{7} \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \\ \frac{4}{7} \end{pmatrix} \\ &= \left(\frac{1}{7}, -\frac{5}{14}, \frac{3}{7}\right)^T. \end{aligned}$$

Exercise 6: Using the Newton formula $x_{n+1} = x_n - f(x)/f'(x)$, show that for $f(x) = \cos(x) - x = 0$, with $x_0 = \pi/4$, that after two iterations the approximation, x_2, to the solution is

- ○ 0.7854
- ○ 0.0167
- ○ 0.8489
- ○ −0.7395
- ○ −0.7489
- ☒ 0.7391

For $f(x) = \cos(x) - x$, then the derivative is $f' = -\sin(x) - 1$. Thus

$$x_{n+1} = x_n + \frac{\cos(x) - x}{\sin(x) + 1}.$$

Substituting the first values in $x_0 = \pi/4 = 0.7853981635$, then yields

$$x_1 = \frac{\pi}{4} + \frac{\cos(\pi/4) - \pi/4}{\sin(\pi/4) + 1}.$$

where $\cos(x_0) = \sin(x_0) = \dfrac{\sqrt{2}}{2}$, so the first approximation is

$$\begin{aligned} x_1 &= \frac{\pi}{4} + \frac{\frac{\sqrt{2}}{2} - \frac{\pi}{4}}{\frac{\sqrt{2}}{2} + 1} \\ &= \frac{\pi}{4} + \frac{\frac{2\sqrt{2}}{4} - \frac{\pi}{4}}{\frac{2\sqrt{2}}{4} + \frac{4}{4}} \\ &= \frac{1}{4}\left(\pi + \frac{2\sqrt{2} - \pi}{2\sqrt{2} + 4}\right) \\ &= 0.7395361337 \end{aligned}$$

Next

$$\begin{aligned} x_2 &= 0.7395361337 + \frac{\cos(0.7395361337) - 0.7395361337}{\sin(0.7395361337) + 1} \\ &= 0.7390851781 \\ &= \boxed{0.7391}. \end{aligned}$$