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CTMS-MAT-13: Numerical Methods

Quiz 2. 13 May 2026

Answer *four* questions only. All questions carry equal marks. Please write you name at the top and please clearly indicate which questions are to be marked.

All trigonometric values are in radians.

Exercise 1: Newton interpolation constructs a interpolating polynomial $p(x)$, using the formula $p = \sum_{i=0}^{n} \alpha_i n_i(x)$, where the basis polynomials are defined as

$$n_0(x) = 1, \quad n_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1}) \quad \text{for} \quad i \geq 1$$

where $p(x_i) = y_i$. Using Newton interpolation on the data:

i	0	1	2
x_i	0	2	4
y_i	2	1	2

find the correct collocation matrix Φ and weighting vector $\vec{\alpha}$ such that $\Phi \vec{\alpha} = \vec{y}$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)^T$ and $\vec{y} = (y_0, y_1, y_2)^T$.

○ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ -1/4 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -1/2 \\ -1/3 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$

○ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Exercise 2: Consider the measurement values $p_0 = 5, p_1 = 4$, and $p_2 = 6$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$.

Let the function $p(u) = a\cos(u) + bu = \sum\limits_{k=0}^{1} \beta_k \varphi_k(u)$ approximate the data in the least squares sense, where $\beta_0 = a$ and $\beta_1 = b$, and $\varphi_0 = \cos(u)$ and $\varphi_1 = u$.

(a) By considering

$$\frac{\partial E}{\partial \beta_j} = \sum_{i=0}^{2} \left(p_i - \sum_{k=0}^{1} \beta_k \varphi_k(u_i) \right) \varphi_j(u_i) = 0,$$

show that the normal equations are given by

$$a \sum_{i=0}^{2} \cos^2(u_i) + b \sum_{i=0}^{2} u_i \cos(u_i) = \sum_{i=0}^{2} p_i \cos(u_i),$$
$$a \sum_{i=0}^{2} u_i \cos(u_i) + b \sum_{i=0}^{2} u_i^2 = \sum_{i=0}^{2} p_i u_i.$$

(b) Write this as a linear equation and solve for a and b, showing

$$a = \left(25 + 2\sqrt{2}\right)/7 \quad \text{and} \quad b = \left(88 - 10\sqrt{2}\right)/(7\pi).$$

(c) Compute numerically the error which is minimized, and show that it can be expressed as

$$E = \frac{2}{7}\left(27 - 10\sqrt{2}\right).$$

Exercise 3: The ordinary differential equation

$$y'(t) = \left(1 - \frac{y}{K}\right)y \quad \text{with} \quad y(0) = 1$$

has the general solution $y(t) = \dfrac{K y_0 e^t}{K + y_0\left(e^t - 1\right)}$. Let $K = 2$, then using the forward Euler method, i.e.

$$u_{n+1} = u_n + hf\left(t_n, u_n\right)$$

with step size $h = 0.1$, calculate the global truncation error, i.e. $|y(2h) - u_2|$, and show that after two steps it is approximately 2.07×10^{-4}.

Exercise 4: Consider the integral

$$I = \int_1^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{\mathrm{d}x}{x} = \ln(2) = 0.6931471805599453.$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2}\left(f\left(x_0\right) + f\left(x_n\right) + 2\sum_{i=1}^{n-1} f\left(x_i\right)\right)$$

where $h = (b-a)/n$ and n is the number of intervals, show that the approximations, I_n, to the integral for $n = 2^k$ where $k = 0$, 1 and 2 are

k	n	I_n
0	1	0.75
1	2	0.70833333
2	4	0.69702381

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \cdots$$

for some constants a_j, and considering the difference between the errors of the Trapezium rule for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3}\left(4R_k^0 - R_{k-1}^0\right)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

Exercise 5: The differential equation

$$y'(t) = 1 - 3y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{3}\left(2e^{-3t} + 1\right)$. From the explicit Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h \sum_{i=1}^{4} b_i f\left(u_k + h \sum_{j=1}^{4} a_{i,j} k_j, t_k + c_i h\right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 0.002
- ○ 0.058
- ○ 0.370
- ○ 0.000115
- ○ 0.965
- ○ 0.000019

Exercise 6: Given the integral

$$I = \int_1^2 \frac{1}{2} + \sin(\pi x)\, \mathrm{d}x$$

what is the error of the approximate integral for the composite Simpson's 1/3 rule

$$S_n = \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n)\right)$$

when using four subintervals?

- ○ 2.30e-5
- ○ 2.67e-6
- ○ 8.44e-7
- ○ 2.11e-4
- ○ 1.45e-3
- ○ 8.30e-2