Dr D. Sinden

CTMS-MAT-13: Numerical Methods

Quiz 2. 13 May 2026

Answer *four* questions only. All questions carry equal marks. Please write you name at the top and please clearly indicate which questions are to be marked.

All trigonometric values are in radians.

Exercise 1: Given the following data:

i	0	1	2
x_i	0	2	4
y_i	2	1	2

Newton interpolation constructs a interpolating polynomial $p(x)$, using the formula $p = \sum_{i=0}^{n} \alpha_i n_i(x)$, where the basis polynomials are defined as

$$n_0(x) = 1, \quad n_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1}) \quad \text{for} \quad i \geq 1$$

where $p(x_i) = y_i$. Using Newton interpolation, which are the correct collocation matrix Φ and weighting vector $\vec{\alpha}$ such that $\Phi\vec{\alpha} = \vec{y}$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)^T$ and $\vec{y} = (y_0, y_1, y_2)^T$.

- [x] $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ -1/4 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -1/2 \\ -1/3 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The Newton basis polynomials are $n_0(x) = 1$, $n_1(x) = x - x_0 = x$, and $n_2(x) = (x - x_0)(x - x_1) = x(x-2)$. The collocation matrix Φ is formed by evaluating $n_j(x_i)$ at each node $x_0 = 0$, $x_1 = 2$, $x_2 = 4$:

$$\begin{aligned} n_0(0) &= 1, \quad n_1(0) = 0, \quad n_2(0) = 0, \\ n_0(2) &= 1, \quad n_1(2) = 2, \quad n_2(2) = 0, \\ n_0(4) &= 1, \quad n_1(4) = 4, \quad n_2(4) = 4(4-2) = 8, \end{aligned}$$

so that

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}.$$

Solving $\Phi\,\vec{\alpha} = \vec{y}$ with $\vec{y} = (2, 1, 2)^T$ by forward substitution: from the first row, $\alpha_0 = 2$; from the second row, $2 + 2\alpha_1 = 1$, giving $\alpha_1 = -\frac{1}{2}$; from the third row, $2 + 4\left(-\frac{1}{2}\right) + 8\alpha_2 = 2$, giving $\alpha_2 = \frac{1}{4}$. Therefore

$$\vec{\alpha} = \begin{pmatrix} 2 \\ -\frac{1}{2} \\ \frac{1}{4} \end{pmatrix}.$$

An alternative approach is simply to multiply the matrices and the vector until the correct answer is found.

Exercise 2: Consider the measurement values $p_0 = 5, p_1 = 4$, and $p_2 = 6$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$.

Let the function $p(u) = a\cos(u) + bu = \sum_{k=0}^{1} \beta_k \varphi_k(u)$ approximate the data in the least squares sense, where $\beta_0 = a$ and $\beta_1 = b$, and $\varphi_0 = \cos(u)$ and $\varphi_1 = u$.

(a) By considering

$$\frac{\partial E}{\partial \beta_j} = \sum_{i=0}^{n} \left(p_i - \sum_{k=0}^{1} \beta_k \varphi_k(u_i) \right) \varphi_j(u_i) = 0,$$

show that the normal equations are given by

$$a \sum_{i=0}^{2} \cos^2(u_i) + b \sum_{i=0}^{2} u_i \cos(u_i) = \sum_{i=0}^{2} p_i \cos(u_i),$$
$$a \sum_{i=0}^{2} u_i \cos(u_i) + b \sum_{i=0}^{2} u_i^2 = \sum_{i=0}^{2} p_i u_i.$$

(b) Write this as a linear equation and solve for a and b, showing

$$a = \left(25 + 2\sqrt{2}\right)/7 \quad \text{and} \quad b = \left(88 - 10\sqrt{2}\right)/(7\pi).$$

(c) Compute numerically the error which is minimized, and show that it can be expressed as

$$E = \frac{2}{7}\left(27 - 10\sqrt{2}\right).$$

(a) We minimize $E = \sum_{i=0}^{2}(p_i - a\cos(u_i) - bu_i)^2$. Setting

$$\frac{\partial E}{\partial \beta_j} = \sum_{i=0}^{2} \left(p_i - \sum_{k=0}^{1} \beta_k \varphi_k(u_i) \right) \varphi_j(u_i) = 0$$

For $j = 0$, $\varphi_0(u_i) = \cos(u_i)$:

$$\sum_{i=0}^{2} (p_i - a\cos(u_i) - bu_i)\cos(u_i) = 0$$
$$\Rightarrow a \sum_{i=0}^{2} \cos^2(u_i) + b \sum_{i=0}^{2} u_i \cos(u_i) = \sum_{i=0}^{2} p_i \cos(u_i).$$

For $j = 1$, $\varphi_1(u_i) = u_i$:

$$\sum_{i=0}^{2} (p_i - a\cos(u_i) - bu_i)\, u_i = 0$$
$$\Rightarrow a \sum_{i=0}^{2} u_i \cos(u_i) + b \sum_{i=0}^{2} u_i^2 = \sum_{i=0}^{2} p_i u_i.$$

(b) We evaluate each sum using $\cos(0) = 1$, $\cos(\pi/4) = \frac{\sqrt{2}}{2}$, $\cos(\pi/2) = 0$:

$$\sum_{i=0}^{2} \cos^2(u_i) = 1 + \tfrac{1}{2} + 0 = \tfrac{3}{2}, \qquad \sum_{i=0}^{2} u_i \cos(u_i) = 0 + \tfrac{\pi\sqrt{2}}{8} + 0 = \tfrac{\pi\sqrt{2}}{8},$$

$$\sum_{i=0}^{2} u_i^2 = 0 + \tfrac{\pi^2}{16} + \tfrac{\pi^2}{4} = \tfrac{5\pi^2}{16}, \qquad \sum_{i=0}^{2} p_i \cos(u_i) = 5 + 2\sqrt{2}, \qquad \sum_{i=0}^{2} p_i u_i = \pi + 3\pi = 4\pi.$$

The linear system is:

$$\begin{pmatrix} \frac{3}{2} & \frac{\pi\sqrt{2}}{8} \\ \frac{\pi\sqrt{2}}{8} & \frac{5\pi^2}{16} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 5 + 2\sqrt{2} \\ 4\pi \end{pmatrix}.$$

The determinant is $\frac{15\pi^2}{32} - \frac{2\pi^2}{64}$, so the inverse is

$$\begin{pmatrix} \frac{5}{7} & -\frac{2\sqrt{2}}{7\pi} \\ -\frac{2\sqrt{2}}{7\pi} & \frac{24}{7\pi^2} \end{pmatrix}$$

Therefore:

$$a = \frac{25 + 2\sqrt{2}}{7}, \qquad b = \frac{88 - 10\sqrt{2}}{7\pi}.$$

(c) Compute the residuals $r_i = p_i - a\cos(u_i) - bu_i$:

$$r_0 = 5 - a = \frac{35 - 25 - 2\sqrt{2}}{7} = \frac{10 - 2\sqrt{2}}{7},$$

$$r_1 = 4 - \frac{a\sqrt{2}}{2} - \frac{b\pi}{4} = 4 - \frac{25\sqrt{2} + 4}{14} - \frac{88 - 10\sqrt{2}}{28} = \frac{112 - 50\sqrt{2} - 8 - 88 + 10\sqrt{2}}{28} = \frac{4 - 10\sqrt{2}}{7},$$

$$r_2 = 6 - \frac{b\pi}{2} = 6 - \frac{88 - 10\sqrt{2}}{14} = \frac{84 - 88 + 10\sqrt{2}}{14} = \frac{5\sqrt{2} - 2}{7}.$$

The minimized error is:

$$\begin{aligned} E &= r_0^2 + r_1^2 + r_2^2 \\ &= \frac{(10 - 2\sqrt{2})^2 + (4 - 10\sqrt{2})^2 + (5\sqrt{2} - 2)^2}{49} \\ &= \frac{(108 - 40\sqrt{2}) + (216 - 80\sqrt{2}) + (54 - 20\sqrt{2})}{49} \\ &= \frac{378 - 140\sqrt{2}}{49} \\ &= \frac{2}{7}\left(27 - 10\sqrt{2}\right). \end{aligned}$$

Exercise 3: The ordinary differential equation

$$y'(t) = \left(1 - \frac{y}{K}\right) y \quad \text{with} \quad y(0) = 1$$

has the general solution $y(t) = \dfrac{K y_0 e^t}{K + y_0 (e^t - 1)}$. Let $K = 2$, then using the forward Euler method, i.e.

$$u_{n+1} = u_n + h f(t_n, u_n)$$

with step size $h = 0.1$, calculate the global truncation error, i.e. $|y(2h) - u_2|$, and show that after two steps it is approximately 2.07×10^{-4}.

The first forward Euler step at $t_0 = 0$, with $u_0 = 1$ is calculated via

$$f(t_0, u_0) = \left(1 - \frac{1}{2}\right)(1) = 0.5,$$

thus, by the forward Euler formula, with $h = 0.1$,

$$\Rightarrow u_1 = 1 + (0.1)(0.5) = 1.05$$

The second step at $t_1 = 0.1$, $u_1 = 1.05$:

$$f(t_1, u_1) = \left(1 - \frac{1.05}{2}\right)(1.05) = (0.475)(1.05) = 0.49875,$$

so that

$$\Rightarrow u_2 = 1.05 + (0.1)(0.49875) = 1.099875.$$

The exact value at $t = 2h = 0.2$ is given by

$$\begin{aligned} y(0.2) &= \frac{2e^{0.2}}{1 + e^{0.2}} \\ &= \frac{2(1.22140276)}{2.22140276} \\ &\approx 1.09966799. \end{aligned}$$

Thus, the global truncation error after two steps is

$$\begin{aligned} |y(2h) - u_2| &= |1.09966799 - 1.099875| \\ &\approx 2.07 \times 10^{-4}. \end{aligned}$$

Exercise 4: Consider the integral

$$I = \int_1^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{\mathrm{d}x}{x} = \ln(2) = 0.6931471805599453$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right)$$

where $h = (b-a)/n$ and n is the number of intervals, show that the approximations to the integral for $n = 2^k$ where $k = 0$, 1 and 2 are

k	n	I_n
0	1	0.75
1	2	0.70833333
2	4	0.69702381

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \cdots$$

for some constants a, and considering the difference between the errors of the Trapezium rule for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3}\left(4R_k^0 - R_{k-1}^0\right)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

(a) For $k = 0$, $n = 1$, $h = 1$, nodes: $x_0 = 1$, $x_1 = 2$:

$$I_1 = \frac{1}{2}(f(1) + f(2)) = \frac{1}{2}\left(1 + \frac{1}{2}\right) = \frac{3}{4} = \boxed{0.75}\,.$$

For $k = 1$, $n = 2$, $h = 1/2$, nodes: $x_0 = 1$, $x_1 = 3/2$, $x_2 = 2$:

$$I_2 = \frac{1}{4}\left(f(1) + 2f\left(\tfrac{3}{2}\right) + f(2)\right) = \frac{1}{4}\left(1 + \frac{4}{3} + \frac{1}{2}\right) = \frac{1}{4}\cdot\frac{17}{6} = \frac{17}{24} = \boxed{0.70833333}\,.$$

For $k = 2$, $n = 4$, $h = 1/4$, nodes: $x_0 = 1$, $x_1 = 5/4$, $x_2 = 3/2$, $x_3 = 7/4$, $x_4 = 2$:

$$I_4 = \frac{1}{8}\left(f(1) + 2f\left(\tfrac{5}{4}\right) + 2f\left(\tfrac{3}{2}\right) + 2f\left(\tfrac{7}{4}\right) + f(2)\right) = \frac{1}{8}\left(1 + \frac{8}{5} + \frac{4}{3} + \frac{8}{7} + \frac{1}{2}\right)$$
$$= \frac{1}{8}\cdot\frac{210 + 336 + 280 + 240 + 105}{210} = \frac{1171}{1680} = \boxed{0.69702381}\,.$$

(b) The error expansion for the Trapezium rule with step size h gives

$$I = I_h + a_1 h^2 + a_2 h^4 + \cdots$$

Replacing h by $h/2$:

$$I = I_{h/2} + a_1\frac{h^2}{4} + a_2\frac{h^4}{16} + \cdots$$

Computing $3I = 4I_{h/2} - I_h$ to eliminate the leading error term, a_1

$$4I - I = 4I_{h/2} - I_h + a_2 h^4\left(\frac{4}{16} - 1\right) + \cdots$$
$$3I = 4I_{h/2} - I_h + \mathcal{O}\left(h^4\right)$$
$$\Rightarrow I = \frac{4I_{h/2} - I_h}{3} + \mathcal{O}\left(h^4\right).$$

Identifying $R^0_{k-1} = I_h$ and $R^0_k = I_{h/2}$ (since doubling n halves h), we obtain the Romberg formula:

$$R^1_k = \frac{1}{3}\left(4R^0_k - R^0_{k-1}\right).$$

(c) Using $R^0_1 = I_2 = \frac{17}{24}$ and $R^0_2 = I_4 = \frac{1171}{1680}$:

$$R^1_2 = \frac{1}{3}\left(4 \cdot \frac{1171}{1680} - \frac{17}{24}\right) = \frac{1}{3}\left(\frac{4684}{1680} - \frac{1190}{1680}\right) = \frac{1}{3} \cdot \frac{3494}{1680} = \frac{1747}{2520} = \boxed{0.693253}.$$

Exercise 5: The differential equation

$$y'(t) = 1 - 3y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{3}\left(2e^{-3t} + 1\right)$. From the explicit Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h\sum_{i=1}^{4} b_i f\left(u_k + h\sum_{j=1}^{4} a_{i,j}k_j, t_k + c_i h\right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 0.002
- ○ 0.058
- ○ 0.370
- ○ 0.000115
- ○ 0.965
- ☒ 0.000019

Given $y'(t) = f(t, y) = 1 - 3y$, $y(0) = 1$, $h = 0.1$, and the classical RK4 method with stages

$$k_1 = f(u_n), \quad k_2 = f\left(u_n + \tfrac{h}{2}k_1\right), \quad k_3 = f\left(u_n + \tfrac{h}{2}k_2\right), \quad k_4 = f(u_n + h\,k_3),$$

$$u_{n+1} = u_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

Step 1: $t_0 = 0$, $u_0 = 1$.

$$\begin{aligned} k_1 &= 1 - 3(1) = -2 \\ k_2 &= 1 - 3\left(1 + \frac{1}{20}(-2)\right) = 1 - 3\left(\frac{9}{10}\right) = -\frac{17}{10} \\ k_3 &= 1 - 3\left(1 + \frac{1}{20}\left(-\frac{17}{10}\right)\right) = 1 - 3\left(\frac{183}{200}\right) = -\frac{349}{200} \\ k_4 &= 1 - 3\left(1 + \frac{1}{10}\left(-\frac{349}{200}\right)\right) = 1 - 3\left(\frac{1651}{2000}\right) = -\frac{2953}{2000} \end{aligned}$$

The weighted sum is

$$k_1 + 2k_2 + 2k_3 + k_4 = -\frac{4000}{2000} - \frac{6800}{2000} - \frac{6980}{2000} - \frac{2953}{2000} = -\frac{20733}{2000}.$$

Therefore

$$u_1 = 1 + \frac{1}{60}\left(-\frac{20733}{2000}\right) = 1 - \frac{20733}{120000} = \frac{33089}{40000} = 0.827225.$$

Step 2: $t_1 = 0.1$, $u_1 = 0.827225$.

$$
\begin{aligned}
k_1 &= 1 - 3 \times 0.827225 = 1 - 2.481675 = -1.481675 \\
k_2 &= 1 - 3\left(0.827225 + \frac{1}{20}(-1.481675)\right) = 1 - 3\,(0.827225 - 0.0740838) = 1 - 3\,(0.7531412) \\
&= 1 - 2.2594237 = -1.2594237 \\
k_3 &= 1 - 3\left(0.827225 + \frac{1}{20}(-1.2594237)\right) = 1 - 3\,(0.827225 - 0.06297119) = 1 - 3\,(0.76425381) \\
&= 1 - 2.29276143 = -1.29276143 \\
k_4 &= 1 - 3\left(0.827225 + \frac{1}{10}(-1.29276143)\right) = 1 - 3\,(0.827225 - 0.129276143) = 1 - 3\,(0.697948857) \\
&= 1 - 2.093846571 = -1.093846571
\end{aligned}
$$

$$
\begin{aligned}
k_1 + 2k_2 + 2k_3 + k_4 &= -1.481675 + 2(-1.2594237) + 2(-1.29276143) + (-1.093846571) \\
&= -1.481675 - 2.5188474 - 2.58552286 - 1.093846571 \\
&= -7.679891831
\end{aligned}
$$

Therefore

$$
\begin{aligned}
u_2 &= 0.827225 - \frac{1}{60}7.679891831 \\
&= 0.827225 - 0.127998 \\
&= 0.69922680094
\end{aligned}
$$

Exact solution:

$$
y(0.2) = \frac{1}{3}\left(2e^{-3/5} + 1\right) = 0.69920775740\ldots
$$

Global truncation error:

$$
|y(2h) - u_2| = |0.69920775740 - 0.69922680094| \approx \boxed{1.904 \times 10^{-5}}\,.
$$

Exercise 6: Given the integral

$$I = \int_1^2 \frac{1}{2} + \sin(\pi x)\, dx$$

what is the error of the approximate integral for Simpson's rule when using four subintervals?

- ○ 2.30e-5
- ○ 2.67e-6
- ○ 8.44e-7
- ○ 2.11e-4
- ☒ 1.45e-3
- ○ 8.30e-2

With $n = 4$ on $[1, 2]$, the step size is

$$h = \frac{2-1}{4} = \frac{1}{4}.$$

The nodes are $x_0 = 1$, $x_1 = 1.25$, $x_2 = 1.5$, $x_3 = 1.75$, $x_4 = 2$.

Evaluating $f(x) = \frac{1}{2} + \sin(\pi x)$ at each node gives

$$f(1) = \frac{1}{2}, \quad f(1.25) = \frac{1}{2} - \frac{\sqrt{2}}{2}, \quad f(1.5) = -\frac{1}{2}, \quad f(1.75) = \frac{1}{2} - \frac{\sqrt{2}}{2}, \quad \text{and} \quad f(2) = \frac{1}{2}.$$

Applying Simpson's rule yields

$$\begin{aligned} S_4 &= \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right] \\ &= \frac{1}{12}\left[\frac{1}{2} + 4\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) + 2\left(-\frac{1}{2}\right) + 4\left(\frac{1}{2} - \frac{\sqrt{2}}{2}\right) + \frac{1}{2}\right] \\ &= \frac{1}{12}\left(4 - 4\sqrt{2}\right) \\ &= \frac{1-\sqrt{2}}{3}. \end{aligned}$$

The exact value of the integral is

$$\begin{aligned} I &= \left[\frac{x}{2} - \frac{\cos(\pi x)}{\pi}\right]_1^2 \\ &= \left(1 - \frac{1}{\pi}\right) - \left(\frac{1}{2} + \frac{1}{\pi}\right) \\ &= \frac{1}{2} - \frac{2}{\pi}. \end{aligned}$$

Hence the error is

$$\begin{aligned} E &= |I - S_4| \\ &= \left|\left(\frac{1}{2} - \frac{2}{\pi}\right) - \frac{1-\sqrt{2}}{3}\right| \\ &= \left|\frac{1}{6} + \frac{\sqrt{2}}{3} - \frac{2}{\pi}\frac{\pi(1+2\sqrt{2}) - 12}{6\pi}\right| \\ &\approx 0.00145 \\ &= 1.45\text{e-}3. \end{aligned}$$