Constructor University **Spring Semester 2025**

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CTMS-MAT-13: Numerical Methods

Problem Sheet 1 Solutions. Released: 19 February 2026

Exercise 1: Let $f(x) = \sin(\omega x)$ with some positive, real number ω.

a) Write down the Taylor polynomials $p_n(x)$ of degree n for $f(x)$ around $c = 0$ for each of the following cases $n = 1, 2, 3, 4$.

b) How large should n be so that $|\sin(x) - p_n(x)| < \varepsilon$ when $\varepsilon = 10^{-4}$ everywhere in the interval $[0, 1]$?

a) To find the Taylor polynomials around $c = 0$, compute the derivatives:

$$f'(x) = \omega \cos(\omega x)$$

Thus, given that around $c = 0$, the contributions from even derivatives will be zero, as $f^{(2n)} = \omega^{2n} \sin(0) = 0$, only odd terms appear in the polynomial expansion

$$p_1(x) = \omega x$$

$$p_2(x) = \omega x$$

$$p_3(x) = \omega x - \frac{\omega^3}{6} x^3$$

$$p_4(x) = \omega x - \frac{\omega^3}{6} x^3$$

b) Finding n for desired accuracy requires that $|\sin(x) - p_n(x)| < \varepsilon = 10^{-4}$ for all $x \in [0, 1]$.

The error in the Taylor polynomial is bounded by:

$$|R_n(x)| \leq \frac{\max_{\xi \in [0,1]} |f^{(n+1)}(\xi)|}{(n+1)!} |x|^{n+1}$$

Since all derivatives of $\sin(x)$ satisfy $|f^{(n+1)}(\xi)| \leq 1$, for $x \in [0, 1]$:

$$|R_n(x)| \leq \frac{1}{(n+1)!}$$

Hence,

$$\frac{1}{(n+1)!} < 10^{-4}$$

Computing factorials:

- $7! = 5040 \Rightarrow \frac{1}{7!} \approx 0.000198 > 10^{-4}$
- $8! = 40320 \Rightarrow \frac{1}{8!} \approx 0.0000248 < 10^{-4}$

Thus, as $7 < n + 1 < 8$, so polynomial with $n = 7$ is required.

Exercise 2:

a) Show that up to quadratic terms, the Taylor series for $\sqrt{1+x}$ about $c = 0$ can be written as:

$$p_2(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2.$$

b) The remainder term is given by

$$\frac{1}{3!}(x-c)^3 f'''(\xi_x),$$

where ξ_x is between x and $c = 0$. Consider the maximum value of the remainder for all $x \in [0,1]$, and, by simplifying, show that the upper bound to the remainder term is 1/16.

c) Considering $x \in \left[0, \frac{1}{2}\right]$, what is the maximum value of the remainder term?

a) Let $f(x) = \sqrt{1+x} = (1+x)^{1/2}$ and compute the derivatives at $c = 0$

$$\begin{aligned}
f(x) &= (1+x)^{1/2} && \Rightarrow & f(0) &= 1 \\
f'(x) &= \frac{1}{2}(1+x)^{-1/2} && \Rightarrow & f'(0) &= \frac{1}{2} \\
f''(x) &= \frac{1}{2}\cdot\left(-\frac{1}{2}\right)(1+x)^{-3/2} = -\frac{1}{4}(1+x)^{-3/2} && \Rightarrow & f''(0) &= -\frac{1}{4}
\end{aligned}$$

The Taylor polynomial of degree 2 is:

$$\begin{aligned}
p_2(x) &= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \\
&= 1 + \frac{1}{2}x + \frac{-1/4}{2}x^2 \\
&= \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2}
\end{aligned}$$

b) The compute the upper bound on the remainder for $x \in [0,1]$, first compute the third derivative:

$$\begin{aligned}
f''(x) &= -\frac{1}{4}(1+x)^{-3/2} \\
f'''(x) &= -\frac{1}{4}\cdot\left(-\frac{3}{2}\right)(1+x)^{-5/2} = \frac{3}{8}(1+x)^{-5/2}
\end{aligned}$$

Thus, the remainder term is given by:

$$\begin{aligned}
R_2(x) &= \frac{1}{3!}(x-0)^3 f'''(\xi_x) \\
&= \frac{1}{6}x^3 \cdot \frac{3}{8}(1+\xi_x)^{-5/2} \\
&= \frac{1}{16}x^3(1+\xi_x)^{-5/2}
\end{aligned}$$

where $\xi_x \in (0,x)$ for $x \in [0,1]$.

To find the maximum of $|R_2(x)|$ for $x \in [0,1]$ note that it is required that both terms are maximized, and that

- $|x^3|$ is maximized when $x = 1$, giving $x^3 = 1$
- $(1+\xi_x)^{-5/2}$ is maximized when $(1+\xi_x)$ is minimized, i.e., when $\xi_x = 0$, giving $(1+0)^{-5/2} = 1$

Therefore:

$$|R_2(x)| \le \frac{1}{16}\cdot 1 \cdot 1 = \boxed{\frac{1}{16}}.$$

c) Maximum remainder for $x \in [0, \frac{1}{2}]$ follows the same logic as before, that is for $x \in \left[0, \frac{1}{2}\right]$, then $\xi_x \in \left[0, \frac{1}{2}\right]$ and the same expression for the remainder is maximized.

$$|R_2(x)| = \frac{1}{16}|x^3|(1 + \xi_x)^{-5/2}$$

Thus

- $|x^3|$ is maximized at $x = \frac{1}{2} \quad \rightarrow \quad x^3 = \frac{1}{8}$
- $(1 + \xi_x)^{-5/2}$ is maximized at $\xi_x = 0 \quad \rightarrow \quad (1 + 0)^{-5/2} = 1$

Therefore:

$$\max_{x \in [0,1/2]} |R_2(x)| = \frac{1}{16} \cdot \frac{1}{8} \cdot 1 = \boxed{\frac{1}{128}}$$

Exercise 3: Compute the Taylor series for $f(x) = e^{\cos(x)}$ around $c = 0$.

As the expansions about zero are given by

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$$

and

$$e^u = 1 + u + \frac{u^2}{2!} + \frac{u^3}{3!} + \cdots$$

Let $u = \cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}(x^6)$. Then note that $u - 1 = -\frac{x^2}{2} + \frac{x^4}{24} + \mathcal{O}(x^6)$. Then:

$$\begin{aligned} e^{\cos(x)} &= e^{1+(u-1)} = e \cdot e^{u-1} \\ &= e \cdot e^{-x^2/2 + x^4/24 + \cdots} \end{aligned}$$

expanding the exponential function

$$\begin{aligned} &\approx e\left[1 + \left(-\frac{x^2}{2} + \frac{x^4}{24} + \cdots\right) + \frac{1}{2}\left(-\frac{x^2}{2} + \frac{x^4}{24} + \cdots\right)^2 + \cdots\right] \\ &= e\left[1 - \frac{x^2}{2} + \frac{x^4}{24} + \frac{x^4}{8} + \cdots\right] \\ &= e\left[1 - \frac{x^2}{2} + x^4\left(\frac{1}{24} + \frac{1}{8}\right) + \cdots\right] \\ &= \boxed{e\left[1 - \frac{x^2}{2} + \frac{x^4}{6} + \cdots\right]} \end{aligned}$$

Exercise 4: Convert the following from one base to another and write down your calculations as an expansion:

a) $(140)_{10}$ to $(\ldots)_2$

b) $(10.75)_{10}$ to $(\ldots)_2$

c) $(111.01001)_2$ to $(\ldots)_8$
Hint: consider $(111)_2 + (010)_2 + (010)_2$ to get a three digit representation in base 10, then convert each digit from base 10 to base 8.

a) $(140)_{10}$ to base 2. Applying Horner's method

$$
\begin{aligned}
140 &= 2\cdot 70 + 0 &&\Rightarrow a_8 = 1\\
70 &= 2\cdot 35 + 0 &&\Rightarrow a_7 = 0\\
35 &= 2\cdot 17 + 1 &&\Rightarrow a_6 = 0\\
17 &= 2\cdot 8 + 1 &&\Rightarrow a_5 = 0\\
8 &= 2\cdot 4 + 0 &&\Rightarrow a_4 = 1\\
4 &= 2\cdot 2 + 0 &&\Rightarrow a_3 = 1\\
2 &= 2\cdot 1 + 0 &&\Rightarrow a_2 = 0\\
1 &= 2\cdot 0 + 1 &&\Rightarrow a_1 = 0
\end{aligned}
$$

That is

$$
\begin{aligned}
\boxed{(10001100)_2} &= 1\cdot 2^7 + 0\cdot 2^6 + 0\cdot 2^5 + 0\cdot 2^4 + 1\cdot 2^3 + 1\cdot 2^2 + 0\cdot 2^1 + 0\cdot 2^0\\
&= 128 + 8 + 4\\
&= (140)_{10}
\end{aligned}
$$

b) Convert $(10.75)_{10}$ to base 2, then first consider the integer part $(10)_{10}$:

$$
\begin{aligned}
10 \div 2 &= 5 &&\text{remainder } 0 &&\Rightarrow a_4 = 1\\
5 \div 2 &= 2 &&\text{remainder } 1 &&\Rightarrow a_3 = 0\\
2 \div 2 &= 1 &&\text{remainder } 0 &&\Rightarrow a_2 = 1\\
1 \div 2 &= 0 &&\text{remainder } 1 &&\Rightarrow a_1 = 0
\end{aligned}
$$

Therefore: $(10)_{10} = (1010)_2 = 1\cdot 2^3 + 10\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 = 2 + 8 = 10$. Consider the fractional part $(0.75)_{10}$. We multiply by 2 repeatedly and take the integer part until the remainder part is sufficiently small

$$
\begin{aligned}
0.75 \times 2 &= 1.5 &&\rightarrow \text{integer part } 1\\
0.5 \times 2 &= 1.0 &&\rightarrow \text{integer part } 1
\end{aligned}
$$

In this case the, after two iterates and then two significant figures, the remainder is exactly zero. Therefore: $(0.75)_{10} = 0.5 + 0.25 = 1\cdot 2^{-1} + 1\cdot 2^{-2} = (0.11)_2$. Thus, the combined result is

$$
(10.75)_{10} = \boxed{(1010.11)_2}\,.
$$

Explicitly

$$
\begin{aligned}
(1010.11)_2 &= 1\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 0\cdot 2^0 + 1\cdot 2^{-1} + 1\cdot 2^{-2}\\
&= 8 + 2 + 0.5 + 0.25\\
&= (10.75)_{10}
\end{aligned}
$$

c) Convert $(111.01001)_2$ to base 8. Note $2^3 = 8$, so to convert from binary to octal, group bits in sets of 3, starting from the decimal point.

The integer part is already grouped into three digits, while the fractional part must be padded with a trailing zero $(01001)_2 = (010, 010)_2$.

Convert each group to base 10, then to base 8:

$$
\begin{aligned}
(111)_2 &= 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 4 + 2 + 1 = 7 = (7)_8 \\
(010)_2 &= 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 2 = (2)_8 \\
(010)_2 &= 0 \cdot 2^2 + 1 \cdot 2^1 + 0 \cdot 2^0 = 2 = (2)_8
\end{aligned}
$$

combining:

$$
(111.01001)_2 = \boxed{(7.22)_8}\,.
$$

Checking

$$
\begin{aligned}
(111.01001)_2 &= 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 + 0 \cdot 2^{-1} + 1 \cdot 2^{-2} + 0 \cdot 2^{-3} + 0 \cdot 2^{-4} + 1 \cdot 2^{-5} \\
&= 4 + 2 + 1 + 0.25 + 0.03125 \\
&= 7.28125
\end{aligned}
$$

and

$$
\begin{aligned}
(7.22)_8 &= 7 \cdot 8^0 + 2 \cdot 8^{-1} + 2 \cdot 8^{-2} \\
&= 7 + 0.25 + 0.03125 \\
&= 7.28125
\end{aligned}
$$

Exercise 5: Webcolors can be expressed with six base-16 (hexadecimal) digits (two each for the red, green and blue components, in that order) prefixed with `#`. The hexadecimal format uses sixteen distinct symbols, most often the symbols `0-9` to represent values 0 to 9, and `A-F` (or alternatively `a-f`) to represent values from ten to fifteen.

a) How many separate shades are there in each channel of an RGB triplet and in total?

b) How are black and white written in this format?

c) Convert the hexadecimal colours `#008ce3`, `#00204d` and `#db4f3d` into RGB triplets.

d) CMYK colours encode four channels (cyan, magenta, yellow and black), taking values between 0-100 (inclusive). Are there more possible representations in the CMYK scheme than hexadecimal?

a) There are 16×16 shades for each channel, i.e. 256 and there are three channels which define a colour in this scheme, so, $256^3 = 16,777,216$.

b) Black is written as $(0,0,0)$ in RGB, so as `#000000` whereas white is $(255,255,255)$ in RGB which is `#ffffff`.

c) The Constructor colour MobilityBlue is given as `#008ce3` $= (0\times16+0, 8\times16+12, 14\times16+3) =$ (0, 140, 227).

The Constructor colour NavyBlue is given as `#00204d` $= (0\times16+0, 2\times16+0, 4\times16+13) =$ (0, 32, 77).

Finally, DiversityRed is given as `#db4f3d` $= (13\times16+11, 4\times16+15, 3\times16+13) =$ (219, 79, 61).

d) CMYK: Each channel 0-100 (101 values) and there are 4 channels:

$$101^4 = 104{,}060{,}401 \text{ colors.}$$

Hexadecimal has 16,777,216 colors. Thus CMYK has more possible colors: $104{,}060{,}401 > 16{,}777{,}216$.

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Exercise 6: The *ternary numerical system* uses three as its base. The number $100,000 = 10^5$ in base 10 requires six digits. Its "radix economy" is therefore $10 \times 6 = 60$. For $10,000 = 10^4$ the radix economy is 50.

a) Show that, in base 2, the same number requires 14 digits, so its radix economy is $2 \times 14 = 28$.

b) Show that, in base 3, it requires 9 digits, so its radix economy is $3 \times 9 = 27$.

c) Show that, for $100,000$ in base 2, the same number requires 17 digits, so its radix economy is $2 \times 17 = 34$.

d) Again, for $100,000$, show that, in base 3, the radix economy is 33.

a) There are two approaches: convert $(10000)_{10}$ to base 2 by successive division or, using the information provided, show that $2^{13} < 10^6 < 2^{14}$. The first method yields with *integer division*

$$\begin{aligned}
10000 \div 2 &= 5000 && \text{remainder } 0\\
5000 \div 2 &= 2500 && \text{remainder } 0\\
2500 \div 2 &= 1250 && \text{remainder } 0\\
1250 \div 2 &= 625 && \text{remainder } 0\\
625 \div 2 &= 312 && \text{remainder } 1\\
312 \div 2 &= 156 && \text{remainder } 0\\
156 \div 2 &= 78 && \text{remainder } 0\\
78 \div 2 &= 39 && \text{remainder } 0\\
39 \div 2 &= 19 && \text{remainder } 1\\
19 \div 2 &= 9 && \text{remainder } 1\\
9 \div 2 &= 4 && \text{remainder } 1\\
4 \div 2 &= 2 && \text{remainder } 0\\
2 \div 2 &= 1 && \text{remainder } 0\\
1 \div 2 &= 0 && \text{remainder } 1
\end{aligned}$$

Reading the remainders from bottom to top: $(10000)_{10} = (10011100010000)_2$. Explicitly checking

$$10000 = 2^{13} + 2^{10} + 2^9 + 2^8 + 2^4 = 8192 + 1024 + 512 + 256 + 16$$

This is 14 digits in a binary system (14 bits), as this is powers of two from 0 up to 13. Thus, the radix economy in a binary system is $2 \times 14 = 28$.

b) Because $3^8 < 10^4 < 3^9$, it is expressed with 9 trits (i.e. trinary digits) so the number of digits required is $3 \times 9 = 27$.

c) As $2^{16} < 10^5 < 2^{17}$, so the number of digits required is $2 \times 17 = 34$.

d) $100,000 = (12002011201)_3$ which has 11 digits, so radix economy is $3 \times 11 = 33$. To show that

$$\begin{aligned}
100000 &= 3 \cdot 33333 + 1\\
33333 &= 3 \cdot 11111 + 0\\
11111 &= 3 \cdot 3703 + 2\\
3703 &= 3 \cdot 1234 + 1\\
1234 &= 3 \cdot 411 + 1\\
411 &= 3 \cdot 137 + 0\\
137 &= 3 \cdot 45 + 2\\
45 &= 3 \cdot 15 + 0\\
15 &= 3 \cdot 5 + 0\\
5 &= 3 \cdot 1 + 2\\
1 &= 3 \cdot 0 + 1
\end{aligned}$$

Reading remainders upward gives the base-3 digits:

$$100000_{10} = (12002011201)_3.$$

Explicitly checking

$$= 1 \cdot 3^{10} + 2 \cdot 3^9 + 0 \cdot 3^8 + 0 \cdot 3^7 + 2 \cdot 3^6 + 0 \cdot 3^5 + 1 \cdot 3^4 + 1 \cdot 3^3 + 2 \cdot 3^2 + 0 \cdot 3 + 1.$$

Another approach is to show that $3^{10} < 10^5 < 3^{11}$.