

## CTMS-MAT-13: Numerical Methods

Problem Sheet 3 Solutions. Released: 19 March 2025

**Exercise 1:** For  $f(x) = x^3 - 7x^2 + 14x - 8$ , note that  $x^* = 1$  is a root.

- a) Given this, find the intervals for the other two roots for which the Bisection method will work.
- b) With  $x_0 = 0.9$  and  $x_1 = 1.2$ , apply three steps of the bisection, Newton's and secant method. (For Newton's method use  $x_0 = 0.9$  as the initial guess)
- c) Compare the errors of the results you computed in (b).

a) For  $f(x) = x^3 - 7x^2 + 14x - 8 = (x - 1)(ax^2 + bx + c)$ , then equating co-efficients leads to  $a = 1, b = 6, c = 8$ . This quadratic can then be factorised to show that the  $f(x) = (x - 1)(x - 2)(x - 4)$ . Thus intervals which contain the other roots must be  $(a, b)$  where  $a < 1$  and  $1 < b < 2$

b) Bisection on  $[0.9, 1.2]$ :

$$m_1 = \frac{0.9+1.2}{2} = 1.05, \quad f(m_1) = 0.140125 > 0 \Rightarrow [0.9, 1.05],$$

$$m_2 = \frac{0.9+1.05}{2} = 0.975, \quad f(m_2) = -0.077515625 < 0 \Rightarrow [0.975, 1.05],$$

$$m_3 = \frac{0.975+1.05}{2} = 1.0125, \quad f(m_3) = 0.036876953125 > 0 \Rightarrow [0.975, 1.0125].$$

Bisection iterates:  $m_1 = 1.050000, m_2 = 0.975000, m_3 = 1.012500$

Newton (start  $x_0 = 0.9$ ). Derivative:  $f'(x) = 3x^2 - 14x + 14$ .

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.9 - \frac{-0.341}{3.83} \approx 0.9890339426,$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \approx 0.9998433806,$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 0.9999999670.$$

Newton:  $x_1 \approx 0.9890339426, x_2 \approx 0.9998433806, x_3 \approx 0.9999999670$

Secant (start  $x_0 = 0.9, x_1 = 1.2$ ). Three iterates of the secant method will produce  $x_2, x_3$  and  $x_4$ .

$$x_2 = x_1 - \frac{f(x_1)(x_1 - x_0)}{f(x_1) - f(x_0)} = 1.2 - \frac{0.448 \cdot 0.3}{0.448 - (-0.341)} = 1.2 - \frac{224}{1315} = \frac{1354}{1315} \approx 1.029657795,$$

$$x_3 = x_2 - \frac{f(x_2)(x_2 - x_1)}{f(x_2) - f(x_1)} \approx 0.989491487,$$

$$x_4 = x_3 - \frac{f(x_3)(x_3 - x_2)}{f(x_3) - f(x_2)} \approx 1.000425494.$$

Secant:  $x_2 \approx 1.029657795, x_3 \approx 0.989491487, x_4 \approx 1.000425494$

b) After three iterates:

$$\text{Bisection: } |m_3 - 1| = 0.0125,$$

$$\text{Newton: } |x_3 - 1| \approx 3.3 \times 10^{-8},$$

$$\text{Secant (at } x_4): |x_4 - 1| \approx 4.25 \times 10^{-4}.$$

**Exercise 2:** For

$$f(x) = x^4 - 8x^2 - 1$$

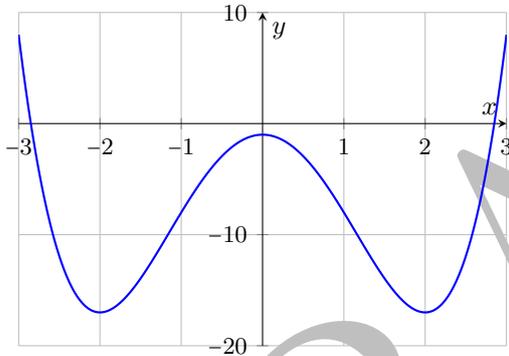
- Draw the function and sketch Newton's method
- With a guess of  $x_0 = 1.5$  show that the second iterate is  $x_2 \approx -0.2788$ .
- With a guess of  $x_0 = 3$  show that the second iterate is  $x_2 \approx 2.850340$ .

a)  $f(x) = x^4 - 8x^2 - 1$  is even. Setting  $y = x^2$  gives  $y^2 - 8y - 1 = 0$ , so the real roots are

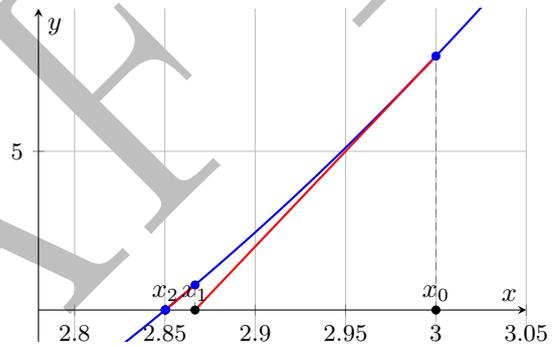
$$x = \pm\sqrt{4 + \sqrt{17}} \approx \pm 2.851.$$

There are two complex roots as well.

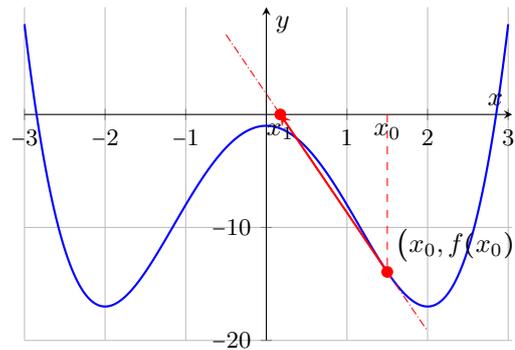
The function is even as  $f(-x) = f(x)$ . Also has  $f(0) = -1 < 0$ , and  $f(x) \rightarrow +\infty$  as  $|x| \rightarrow \infty$ , hence it crosses the  $x$ -axis at the symmetric roots  $\pm\sqrt{4 + \sqrt{17}}$ . Newton's method tangents from a positive  $x_0 \in (0, 2)$  first move toward the origin, then cross to the negative side before converging to  $-\sqrt{4 + \sqrt{17}}$ ; similarly for symmetric starts.



Zoom near  $x_0 = 3$  with Newton steps



Zoom near  $x_0 = 1.5$  with Newton steps



b) Newton's method:  $f'(x) = 4x^3 - 16x$  and  $x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$  with  $x_0 = 1.5$ .

$$\begin{aligned}x_1 &= 1.5 - \frac{1.5^4 - 8(1.5)^2 - 1}{4(1.5)^3 - 16(1.5)} \\&= 1.5 - \frac{-223/16}{-21/2} \\&= \frac{29}{168} \\&\approx 0.172619,\end{aligned}$$

$$x_2 = x_1 - \frac{x_1^4 - 8x_1^2 - 1}{4x_1^3 - 16x_1} \approx -0.2788.$$

c) As before, but with  $x_0 = 3$ , so  $f(x_0) = 8$  and  $f'(x_0) = 60$

$$\begin{aligned}x_1 &= 3 - \frac{8}{60} \\&= \frac{43}{15} \\&\approx 2.8666667,\end{aligned}$$

$$x_2 = x_1 - \frac{x_1^4 - 8x_1^2 - 1}{4x_1^3 - 16x_1} \approx 2.8503395.$$

**Exercise 3:** Starting with  $(x_0, y_0) = (0, 0)$  apply two iterations of the Newton method for the system of non-linear equations

$$\begin{aligned} -x^2 + (x-1)(y+1) + 4y^3 &= 5 \\ (x-2)^2 + (3y-2)^2 &= 5 \end{aligned}$$

to find  $(x_2, y_2)$ .

Write the system as  $F(x, y) = \vec{0}$  with

$$\begin{aligned} F_1(x, y) &= -x^2 + (x-1)(y+1) + 4y^3 - 5, \\ F_2(x, y) &= (x-2)^2 + (3y-2)^2 - 5, \end{aligned}$$

with the Jacobian

$$J(x, y) = \begin{pmatrix} -2x + y + 1 & x - 1 + 12y^2 \\ 2(x-2) & 6(3y-2) \end{pmatrix}.$$

Start at  $(x_0, y_0) = (0, 0)$ . Then

$$F(0, 0) = \begin{pmatrix} -6 \\ 3 \end{pmatrix}, \quad J(0, 0) = \begin{pmatrix} 1 & -1 \\ -4 & -12 \end{pmatrix}.$$

Solve  $J(0, 0) \Delta_0 = -F(0, 0)$ , i.e, solve the system given by  $J(x_k)(x_{k+1} - x_k) = -f(x_k)$ , with  $\Delta_k = x_{k+1} - x_k$ . Thus

$$\begin{cases} \Delta x - \Delta y = 6, \\ -4\Delta x - 12\Delta y = -3, \end{cases} \Rightarrow \Delta y = -\frac{21}{16}, \quad \Delta x = \frac{75}{16}.$$

Hence the updated iterates are

$$(x_1, y_1) = \left( \frac{75}{16}, -\frac{21}{16} \right) = (4.6875, -1.3125).$$

Next, evaluate at  $(x_1, y_1)$ :

$$\begin{aligned} F_1(x_1, y_1) &= -\frac{5625}{256} - \frac{295}{256} - \frac{9261}{1024} - 5 \\ &= -\frac{38061}{1024}, \\ F_2(x_1, y_1) &= \frac{1849}{256} + \frac{9025}{256} - 5 \\ &= \frac{9594}{256}, \\ J(x_1, y_1) &= \begin{pmatrix} -\frac{155}{16} & \frac{1559}{64} \\ \frac{43}{8} & -\frac{285}{8} \end{pmatrix}. \end{aligned}$$

Solve  $J(x_1, y_1) \Delta_1 = -F(x_1, y_1)$ , by for example, Gaussian elimination, so that one finds

$$\Delta x = -\frac{1,684,431}{877,304} \approx -1.920008, \quad \Delta y = \frac{1,337,517}{1,754,608} \approx 0.762288.$$

Hence

$$(x_2, y_2) = (x_1 + \Delta x, y_1 + \Delta y) \approx (2.767492, -0.550212).$$

**Exercise 4:** Consider the linear system  $A\mathbf{x} = \mathbf{b}$ , with

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 2 & -3 \\ 0 & 4 & 4 \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix}.$$

a) Find the inverse to the matrix using Gaussian elimination.

b) Show that after two iterations of the Jacobi scheme, with an initial guess  $\mathbf{x}_0 = (1, 1, 1)^T$  the error

$$\|\mathbf{x}^* - \mathbf{x}_2\|_2 = \|(65/12, 139/24, 5/12)\|_2 = \frac{\sqrt{36321}}{24} \approx 7.94086.$$

a) Augment  $[A|I]$  and reduce:

$$\left[ \begin{array}{ccc|ccc} 1 & 0 & 5 & 1 & 0 & 0 \\ 2 & 2 & -3 & 0 & 1 & 0 \\ 0 & 4 & 4 & 0 & 0 & 1 \end{array} \right] \rightarrow \left[ \begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ 0 & 1 & 0 & -\frac{2}{15} & \frac{1}{15} & \frac{13}{60} \\ 0 & 0 & 1 & \frac{2}{15} & -\frac{1}{15} & \frac{1}{30} \end{array} \right].$$

Thus

$$A^{-1} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{2}{15} & \frac{1}{15} & \frac{13}{60} \\ \frac{2}{15} & -\frac{1}{15} & \frac{1}{30} \end{pmatrix}$$

and the exact solution is  $\mathbf{x}^* = A^{-1}\mathbf{b} = (\frac{1}{6}, \frac{25}{12}, \frac{1}{6})^T$ .

b) Jacobi updates, with

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

So that

$$I - D^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & -\frac{1}{6} \\ -\frac{2}{15} & \frac{1}{15} & \frac{13}{60} \\ \frac{2}{15} & -\frac{1}{15} & \frac{1}{30} \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{5}{6} \\ -1 & 0 & \frac{3}{2} \\ 0 & -1 & 0 \end{pmatrix}.$$

and

$$D^{-1}\mathbf{b} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 9 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 9/4 \end{pmatrix}$$

First iterate:

$$\begin{aligned} \mathbf{x}^{(1)} &= (I - D^{-1}A)\mathbf{x}^{(0)} + D^{-1}\mathbf{b} \\ &= \begin{pmatrix} 0 & 0 & -\frac{5}{6} \\ -1 & 0 & \frac{3}{2} \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 9/4 \end{pmatrix} \\ &= \begin{pmatrix} -5/6 \\ 1/2 \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ 9/4 \end{pmatrix} \\ &= \begin{pmatrix} -4/6 \\ 5/2 \\ 5/4 \end{pmatrix} \end{aligned}$$

Similarly:

$$\mathbf{x}^{(2)} = (-5.25, 7.875, -0.25)^T.$$

Hence

$$\mathbf{e} = \mathbf{x}^* - \mathbf{x}^{(2)} = \left( \frac{65}{12}, -\frac{139}{24}, \frac{5}{12} \right)^\top,$$

so

$$\|e^{(2)}\|_2 = \sqrt{\left(\frac{65}{12}\right)^2 + \left(\frac{139}{24}\right)^2 + \left(\frac{5}{12}\right)^2} = \frac{\sqrt{36321}}{24} \approx 7.94086.$$

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