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CTMS-MAT-13: Numerical Methods

Problem Sheet 4 Solutions. Released 21 April 2026

Exercise 1: Consider the function $f(x) = e^{x+1}$ and the set of equally spaced nodes $0, 1/3, 2/3, 1$. The Newton form of a polynomial interpolant is given by

$$p_n(x) = \sum_{i=0}^{n} a_i n_i(x)$$

where the first polynomial is given by $n_0(x) = 1$ and

$$n_i(x) = (x - x_0)(x - x_1)\ldots(x - x_{i-1}) \quad \text{for} \quad i > 0.$$

Derive the polynomial $p_n(x)$ in Newton form that interpolates $f(x)$ at the four given nodes.

Interpolate $f(x) = e^{x+1}$ at the four equally spaced nodes

$$x_0 = 0, \quad x_1 = \tfrac{1}{3}, \quad x_2 = \tfrac{2}{3}, \quad x_3 = 1.$$

The function takes the four values

$$f(x_0) = e, \quad f(x_1) = e^{4/3}, \quad f(x_2) = e^{5/3}, \quad f(x_3) = e^2.$$

The three first-order divided differences are given by

$$\begin{aligned} f[x_0, x_1] &= \frac{e^{4/3} - e}{1/3} = 3\left(e^{4/3} - e\right), \\ f[x_1, x_2] &= \frac{e^{5/3} - e^{4/3}}{1/3} = 3\left(e^{5/3} - e^{4/3}\right), \\ f[x_2, x_3] &= \frac{e^2 - e^{5/3}}{1/3} = 3\left(e^2 - e^{5/3}\right). \end{aligned}$$

The two second-order divided differences given by

$$\begin{aligned} f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{3(e^{5/3} - e^{4/3}) - 3(e^{4/3} - e)}{2/3} = \frac{9}{2}\left(e^{5/3} - 2e^{4/3} + e\right), \\ f[x_1, x_2, x_3] &= \frac{f[x_2, x_3] - f[x_1, x_2]}{x_3 - x_1} = \frac{3(e^2 - e^{5/3}) - 3(e^{5/3} - e^{4/3})}{2/3} = \frac{9}{2}\left(e^2 - 2e^{5/3} + e^{4/3}\right) \end{aligned}$$

and, finally, the one third-order divided difference:

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0} = \frac{9}{2}\left(e^2 - 3e^{5/3} + 3e^{4/3} - e\right).$$

The Newton basis polynomials are

$$\begin{aligned} n_0(x) &= 1, \\ n_1(x) &= x, \\ n_2(x) &= x\left(x - \tfrac{1}{3}\right), \\ n_3(x) &= x\left(x - \tfrac{1}{3}\right)\left(x - \tfrac{2}{3}\right). \end{aligned}$$

The coefficients are the leading divided differences:

$$a_0 = e, \quad a_1 = 3(e^{4/3} - e), \quad a_2 = \tfrac{9}{2}(e^{5/3} - 2e^{4/3} + e), \quad a_3 = \tfrac{9}{2}(e^2 - 3e^{5/3} + 3e^{4/3} - e).$$

Therefore, the cubic interpolating polynomial in Newton form is:

$$p_3(x) = e + 3(e^{4/3} - e)\,x + \frac{9}{2}(e^{5/3} - 2e^{4/3} + e)\,x\left(x - \tfrac{1}{3}\right) + \frac{9}{2}(e^2 - 3e^{5/3} + 3e^{4/3} - e)\,x\left(x - \tfrac{1}{3}\right)\left(x - \tfrac{2}{3}\right).$$

Exercise 2: Derive the four Lagrange polynomials for $f(x) = e^{x+2}$ evaluated at the set of equally spaced nodes $0, 1/3, 2/3, 1$.

The four Lagrange basis cubic polynomials for $f(x) = e^{x+2}$ evaluated at the equally spaced nodes

$$x_0 = 0, \quad x_1 = \tfrac{1}{3}, \quad x_2 = \tfrac{2}{3}, \quad x_3 = 1.$$

Recall that the k-th Lagrange basis polynomial is defined as

$$L_k(x) = \prod_{\substack{j=0\\ j\neq k}}^{3} \frac{x - x_j}{x_k - x_j}.$$

For $L_0(x)$

$$L_0(x) = \frac{(x-\frac{1}{3})(x-\frac{2}{3})(x-1)}{(0-\frac{1}{3})(0-\frac{2}{3})(0-1)} = \frac{(x-\frac{1}{3})(x-\frac{2}{3})(x-1)}{-\frac{2}{9}} = -\frac{9}{2}\left(x-\tfrac{1}{3}\right)\left(x-\tfrac{2}{3}\right)(x-1).$$

For $L_1(x)$

$$L_1(x) = \frac{x(x-\frac{2}{3})(x-1)}{(\frac{1}{3})(\frac{1}{3}-\frac{2}{3})(\frac{1}{3}-1)} = \frac{x(x-\frac{2}{3})(x-1)}{(\frac{1}{3})(-\frac{1}{3})(-\frac{2}{3})} = \frac{27}{2}\,x\left(x-\tfrac{2}{3}\right)(x-1).$$

For $L_2(x)$

$$L_2(x) = \frac{x(x-\frac{1}{3})(x-1)}{(\frac{2}{3})(\frac{2}{3}-\frac{1}{3})(\frac{2}{3}-1)} = \frac{x(x-\frac{1}{3})(x-1)}{(\frac{2}{3})(\frac{1}{3})(-\frac{1}{3})} = -\frac{27}{2}\,x\left(x-\tfrac{1}{3}\right)(x-1).$$

For $L_3(x)$

$$L_3(x) = \frac{x(x-\frac{1}{3})(x-\frac{2}{3})}{(1)(1-\frac{1}{3})(1-\frac{2}{3})} = \frac{x(x-\frac{1}{3})(x-\frac{2}{3})}{(\frac{2}{3})(\frac{1}{3})} = \frac{9}{2}\,x\left(x-\tfrac{1}{3}\right)\left(x-\tfrac{2}{3}\right).$$

The interpolating polynomial is then given by

$$p_3(x) = \sum_{k=0}^{3} f(x_k)\,L_k(x) = e^2\,L_0(x) + e^{7/3}\,L_1(x) + e^{8/3}\,L_2(x) + e^3\,L_3(x).$$

Collecting terms by powers of x yields

$$\begin{aligned} p_3(x) = &\left(-\tfrac{9}{2}e^2 + \tfrac{27}{2}e^{7/3} - \tfrac{27}{2}e^{8/3} + \tfrac{9}{2}e^3\right)x^3 \\ &+ \left(9e^2 - \tfrac{45}{2}e^{7/3} + 18e^{8/3} - \tfrac{9}{2}e^3\right)x^2 \\ &+ \left(-\tfrac{11}{2}e^2 + 9e^{7/3} - \tfrac{9}{2}e^{8/3} + e^3\right)x \\ &+ e^2. \end{aligned}$$

Factoring out $\frac{9}{2}$ from the cubic coefficient gives

$$p_3(x) = e^2 + \left(-\tfrac{11}{2}e^2 + 9e^{7/3} - \tfrac{9}{2}e^{8/3} + e^3\right)x + \tfrac{9}{2}\left(2e^2 - 5e^{7/3} + 4e^{8/3} - e^3\right)x^2 + \tfrac{9}{2}\left(-e^2 + 3e^{7/3} - 3e^{8/3} + e^3\right)x^3,$$

which, upon factoring the cubic coefficient as $\frac{9}{2}(e^3 - 3e^{8/3} + 3e^{7/3} - e^2)$, agrees with the Newton form derived in Exercise 1 with the coefficients

$$a_0 = e^2, \quad a_1 = 3(e^{7/3} - e^2), \quad a_2 = \tfrac{9}{2}(e^{8/3} - 2e^{7/3} + e^2), \quad a_3 = \tfrac{9}{2}(e^3 - 3e^{8/3} + 3e^{7/3} - e^2).$$

Exercise 3:

a) At $t = (1, 2, 3)$ values are measured as $p = (2, 4, 4.5)$. Using Aitken's method, verify that when $t = 1.5$, the interpolated value is approximately $p = 3.875$

$$
\begin{array}{ccc}
p_{0,0} = 2 & & \\
 & p_{0,1} = 3 & \\
p_{1,0} = 4 & & p_{0,2} = 3.875 \\
 & p_{1,1} = 3.75 & \\
p_{2,0} = 4.5 & &
\end{array}
$$

b) For data measured at $t = (1, 2, 3, 5)$, whose values are measured as $p = (2, 4, 4.5, 5)$, use Aitken's method to find the approximate value at $t = 1.5$.

a) Given the data

$$t = (1, 2, 3), \quad p = (2, 4, 4.5),$$

using Aitken's method to interpolate at $t^* = 1.5$ requires computing the terms

$$p_{i,k}(t^*) = \frac{(t^* - t_i)\, p_{i+1,k-1} - (t^* - t_{i+k})\, p_{i,k-1}}{t_{i+k} - t_i}.$$

Starting from the zeroth-order values

$$p_{0,0} = 2, \quad p_{1,0} = 4, \quad p_{2,0} = 4.5,$$

the first-order values are

$$p_{0,1} = \frac{(1.5-1)\cdot 4 - (1.5-2)\cdot 2}{2-1} = \frac{0.5\cdot 4 + 0.5\cdot 2}{1} = 3,$$

$$p_{1,1} = \frac{(1.5-2)\cdot 4.5 - (1.5-3)\cdot 4}{3-2} = \frac{-0.5\cdot 4.5 + 1.5\cdot 4}{1} = 3.75,$$

and the second-order value is

$$p_{0,2} = \frac{(1.5-1)\cdot 3.75 - (1.5-3)\cdot 3}{3-1} = \frac{0.5\cdot 3.75 + 1.5\cdot 3}{2} = \frac{1.875 + 4.5}{2} = \frac{6.375}{2} = 3.875.$$

Hence the interpolated value at $t^* = 1.5$ is $p \approx 3.875$, as required.

b) We are now given the data

$$t = (1, 2, 3, 5), \quad p = (2, 4, 4.5, 5),$$

and we wish to interpolate at $t^* = 1.5$. We again apply Aitken's method. The zeroth-order values are

$$p_{0,0} = 2, \quad p_{1,0} = 4, \quad p_{2,0} = 4.5, \quad p_{3,0} = 5.$$

The first-order values are

$$p_{0,1} = \frac{(1.5-1)\cdot 4 - (1.5-2)\cdot 2}{2-1} = 3,$$

$$p_{1,1} = \frac{(1.5-2)\cdot 4.5 - (1.5-3)\cdot 4}{3-2} = 3.75,$$

$$p_{2,1} = \frac{(1.5-3)\cdot 5 - (1.5-5)\cdot 4.5}{5-3} = \frac{-7.5 + 15.75}{2} = 4.125.$$

The second-order values are

$$p_{0,2} = \frac{(1.5-1)\cdot 3.75 - (1.5-3)\cdot 3}{3-1} = \frac{1.875 + 4.5}{2} = 3.1875,$$

$$p_{1,2} = \frac{(1.5-2)\cdot 4.125 - (1.5-5)\cdot 3.75}{5-2} = \frac{-2.0625 + 13.125}{3} = \frac{11.0625}{3} = 3.6875.$$

The third-order value is

$$p_{0,3} = \frac{(1.5-1)\cdot 3.6875 - (1.5-5)\cdot 3.1875}{5-1} = \frac{1.84375 + 11.15625}{4} = \frac{13}{4} = 3.25.$$

Hence the interpolated value at $t^* = 1.5$ is

$$p(1.5) \approx 3.25.$$

Exercise 4: Consider the measurement values $p_0 = 5, p_1 = 4$, and $p_2 = 6$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$. Let the function $p(u) = \alpha\cos(u) + \beta u$ approximate the data in the least squares sense.

a) Show that the normal equations are given by

$$\alpha\sum_{i=0}^{2}\cos^2(u_i) + \beta\sum_{i=0}^{2}u_i\cos(u_i) = \sum_{i=0}^{2}p_i\cos(u_i),$$
$$\alpha\sum_{i=0}^{2}u_i\cos(u_i) + \beta\sum_{i=0}^{2}u_i^2 = \sum_{i=0}^{2}p_i u_i.$$

b) Solve the normal equations for α and β.

c) Compute the error in the L_2 sense that is minimized in **(b)**.

d) What is the solution and what is the error if the last measurement value is now $p_2 = 5$?

a) The least squares problem minimises

$$E(\alpha,\beta) = \sum_{i=0}^{2}\big(p(u_i) - p_i\big)^2 = \sum_{i=0}^{2}\big(\alpha\cos(u_i) + \beta u_i - p_i\big)^2.$$

Setting the partial derivatives to zero,

$$\frac{\partial E}{\partial\alpha} = 2\sum_{i=0}^{2}\big(\alpha\cos(u_i) + \beta u_i - p_i\big)\cos(u_i) = 0,$$
$$\frac{\partial E}{\partial\beta} = 2\sum_{i=0}^{2}\big(\alpha\cos(u_i) + \beta u_i - p_i\big)u_i = 0,$$

which rearrange to the normal equations

$$\alpha\sum_{i=0}^{2}\cos^2(u_i) + \beta\sum_{i=0}^{2}u_i\cos(u_i) = \sum_{i=0}^{2}p_i\cos(u_i),$$
$$\alpha\sum_{i=0}^{2}u_i\cos(u_i) + \beta\sum_{i=0}^{2}u_i^2 = \sum_{i=0}^{2}p_i u_i.$$

b) The required sums are evaluated at the nodes. Using $\cos(0) = 1$, $\cos(\pi/4) = \frac{\sqrt{2}}{2}$, $\cos(\pi/2) = 0$,

$$\sum_{i=0}^{2}\cos^2(u_i) = 1 + \tfrac{1}{2} + 0 = \tfrac{3}{2},$$
$$\sum_{i=0}^{2}u_i\cos(u_i) = 0\cdot 1 + \tfrac{\pi}{4}\cdot\tfrac{\sqrt{2}}{2} + \tfrac{\pi}{2}\cdot 0 = \frac{\pi\sqrt{2}}{8},$$
$$\sum_{i=0}^{2}u_i^2 = 0 + \tfrac{\pi^2}{16} + \tfrac{\pi^2}{4} = \frac{5\pi^2}{16},$$
$$\sum_{i=0}^{2}p_i\cos(u_i) = 5\cdot 1 + 4\cdot\tfrac{\sqrt{2}}{2} + 6\cdot 0 = 5 + 2\sqrt{2},$$
$$\sum_{i=0}^{2}p_i u_i = 5\cdot 0 + 4\cdot\tfrac{\pi}{4} + 6\cdot\tfrac{\pi}{2} = \pi + 3\pi = 4\pi.$$

The normal equations therefore become

$$\frac{3}{2}\alpha + \frac{\pi\sqrt{2}}{8}\beta = 5 + 2\sqrt{2},$$
$$\frac{\pi\sqrt{2}}{8}\alpha + \frac{5\pi^2}{16}\beta = 4\pi.$$

In matrix form,

$$\begin{pmatrix} \frac{3}{2} & \frac{\pi\sqrt{2}}{8} \\ \frac{\pi\sqrt{2}}{8} & \frac{5\pi^2}{16} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 5+2\sqrt{2} \\ 4\pi \end{pmatrix}.$$

This can be solved to show that the least squares solution is

$$\alpha = \frac{25+2\sqrt{2}}{7} \approx 3.976, \qquad \beta = \frac{2(44-5\sqrt{2})}{7\pi} \approx 3.337.$$

c) The L_2 error is

$$E = \sum_{i=0}^{2} \big(p(u_i) - p_i\big)^2.$$

Evaluating the fitted function at each node,

$$\begin{aligned} p(u_0) &= \alpha\cos(0) + \beta \cdot 0 = \alpha \approx 3.976, \\ p(u_1) &= \alpha \cdot \tfrac{\sqrt{2}}{2} + \beta \cdot \tfrac{\pi}{4} \approx 3.976 \times 0.7071 + 3.337 \times 0.7854 \approx 4.430, \\ p(u_2) &= \alpha \cdot 0 + \beta \cdot \tfrac{\pi}{2} \approx 3.337 \times 1.5708 \approx 5.241. \end{aligned}$$

The residuals are

$$r_0 = 3.976 - 5 = -1.024, \quad r_1 = 4.430 - 4 = 0.430, \quad r_2 = 5.241 - 6 = -0.759,$$

giving

$$E \approx (-1.024)^2 + (0.430)^2 + (-0.759)^2 \approx 1.049 + 0.185 + 0.576 \approx 1.810.$$

d) If instead $p_2 = 5$, the right-hand sides of the normal equations change to

$$\sum_{i=0}^{2} p_i \cos(u_i) = 5 + 2\sqrt{2}, \quad \sum_{i=0}^{2} p_i u_i = \pi + \tfrac{5\pi}{2} = \tfrac{7\pi}{2}.$$

(the first part of the right hand side remains the same). The second equation changes, giving

$$\frac{\pi\sqrt{2}}{8}\alpha + \frac{5\pi^2}{16}\beta = \frac{7\pi}{2}.$$

In matrix form,

$$\begin{pmatrix} \frac{3}{2} & \frac{\pi\sqrt{2}}{8} \\ \frac{\pi\sqrt{2}}{8} & \frac{5\pi^2}{16} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 5+2\sqrt{2} \\ \frac{7\pi}{2} \end{pmatrix}.$$

Solving again by Cramer's rule,

$$\alpha = \frac{25+3\sqrt{2}}{7} \approx 3.976,$$

$$\beta = \frac{16}{7\pi^2}\left(\frac{3}{2}\cdot\frac{7\pi}{2} - \frac{\pi\sqrt{2}}{8}(5+2\sqrt{2})\right) = \frac{16}{7\pi}\left(\frac{21}{4} - \frac{5\sqrt{2}+4}{8}\right) = \frac{16}{7\pi}\cdot\frac{38-5\sqrt{2}}{8} = \frac{2(38-5\sqrt{2})}{7\pi} \approx 2.81285.$$

Evaluating the fitted function at the nodal values, and summing the residuals gives the corresponding L_2 as The residuals and squared residuals are

i	u_i	$p(u_i) = \alpha\cos(u_i) + \beta u_i$	p_i	$r_i = p(u_i) - p_i$	r_i^2
0	0	4.178	5	−0.822	0.676
1	$\pi/4$	4.163	4	+0.163	0.027
2	$\pi/2$	4.418	5	−0.582	0.339

giving

$$E \approx 0.676 + 0.027 + 0.339 \approx 1.042.$$