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CTMS-MAT-13: Numerical Methods

Problem Sheet 5 Solutions. Released 22 April 2026

Exercise 1:

- a) For Gaussian quadrature on $[-1, 1]$ and $n = 3$, show that the four Gauss nodes x_0, \dots, x_3 as the roots of the polynomial $q(x)$ that is orthogonal to $1, x, x^2$ and x^3 are given by

$$x = \pm \sqrt{\frac{15 \pm 2\sqrt{30}}{35}}.$$

- b) To illustrate the full quadrature, determine the first corresponding weight A_0 of the 4 point Gaussian quadrature.
- c) Based on your results of (a), find the corresponding Legendre polynomial that uses the additional condition of $q(1) = 1$.
- d) Consider Gauss quadrature with 2 nodes on the interval $[-1, 1]$. Derive the resulting Legendre polynomial $q(x)$ as a solution to the orthogonality condition:

$$\int_{-1}^1 q(x) x^k dx = 0$$

for $k = 0, 1$. Determine the roots of $q(x)$ to arrive at the Gauss nodes. Use your result to approximate the integral:

$$\int_{-\frac{3\pi}{\sqrt{3}}}^{\frac{3\pi}{\sqrt{3}}} \sin^2(x) \cos(x) dx.$$

Remember to convert into the appropriate interval. You may give the result either as an exact value or use a calculator to arrive at the final number.

- a) We seek $q(x)$ of degree 4 orthogonal to $1, x, x^2, x^3$ on $[-1, 1]$:

$$\int_{-1}^1 q(x) x^k dx = 0, \quad k = 0, 1, 2, 3.$$

Write $q(x) = x^4 + ax^3 + bx^2 + cx + d$.

By symmetry of $[-1, 1]$, odd-degree terms vanish, so $a = c = 0$:

$$q(x) = x^4 + bx^2 + d.$$

Orthogonality to 1:

$$\int_{-1}^1 (x^4 + bx^2 + d) dx = 2 \left(\frac{1}{5} + \frac{b}{3} + d \right) = 0 \Rightarrow \frac{1}{5} + \frac{b}{3} + d = 0. \tag{I}$$

Orthogonality to x^2 :

$$\int_{-1}^1 (x^4 + bx^2 + d) x^2 dx = 2 \left(\frac{1}{7} + \frac{b}{5} + \frac{d}{3} \right) = 0 \Rightarrow \frac{1}{7} + \frac{b}{5} + \frac{d}{3} = 0. \tag{II}$$

Solving (I) and (II). From (I): $d = -\frac{1}{5} - \frac{b}{3}$. Substitute into (II):

$$\frac{1}{7} + \frac{b}{5} + \frac{1}{3} \left(-\frac{1}{5} - \frac{b}{3} \right) = 0 \Rightarrow \frac{1}{7} - \frac{1}{15} + b \left(\frac{1}{5} - \frac{1}{9} \right) = 0.$$

$$\frac{15-7}{105} + b \cdot \frac{9-5}{45} = 0 \Rightarrow \frac{8}{105} + \frac{4b}{45} = 0 \Rightarrow b = -\frac{8}{105} \cdot \frac{45}{4} = -\frac{6}{7}.$$

Then $d = -\frac{1}{5} + \frac{6}{21} = -\frac{1}{5} + \frac{2}{7} = \frac{-7+10}{35} = \frac{3}{35}$.

Hence:

$$q(x) = x^4 - \frac{6}{7}x^2 + \frac{3}{35}.$$

This is a disguised quadratic. Let $u = x^2$, thus

$$\begin{aligned} u^2 - \frac{6}{7}u + \frac{3}{35} = 0 \Rightarrow u &= \frac{\frac{6}{7} \pm \sqrt{\frac{36}{49} - \frac{12}{35}}}{2} \\ &= \frac{\frac{6}{7} \pm \sqrt{\frac{180-84}{245}}}{2} \\ &= \frac{\frac{6}{7} \pm \sqrt{\frac{96}{245}}}{2} \end{aligned}$$

note that

$$\sqrt{\frac{96}{245}} = \frac{4\sqrt{6}}{7\sqrt{5}} = \frac{4\sqrt{30}}{35}$$

thus

$$\begin{aligned} u &= \frac{\frac{30}{35} \pm \frac{4\sqrt{30}}{35}}{2} \\ &= \frac{30 \pm 4\sqrt{30}}{70} \\ &= \frac{15 \pm 2\sqrt{30}}{35}. \end{aligned}$$

Therefore:

$$x = \pm \sqrt{\frac{15 \pm 2\sqrt{30}}{35}}$$

giving four real nodes $x_0 < x_1 < x_2 < x_3$.

b) The weights are determined by:

$$A_i = \int_{-1}^1 \prod_{j \neq i} \frac{x - x_j}{x_i - x_j} dx.$$

For a 4-point rule, all weights satisfy $\sum_{i=0}^3 A_i = 2$ and by symmetry $A_0 = A_3$, $A_1 = A_2$.

The Gaussian quadrature weights satisfy the system obtained by requiring exact integration of $1, x, x^2, x^3$:

$$A_0 + A_1 + A_2 + A_3 = 2, \tag{W1}$$

$$A_0x_0 + A_1x_1 + A_2x_2 + A_3x_3 = 0, \tag{W2}$$

$$A_0x_0^2 + A_1x_1^2 + A_2x_2^2 + A_3x_3^2 = \frac{2}{3}, \tag{W3}$$

$$A_0x_0^3 + A_1x_1^3 + A_2x_2^3 + A_3x_3^3 = 0. \tag{W4}$$

Using symmetry $A_0 = A_3$, $A_1 = A_2$, $x_3 = -x_0$, $x_2 = -x_1$:

From (W1): $2A_0 + 2A_1 = 2 \Rightarrow A_0 + A_1 = 1$.

From (W3): $2A_0x_0^2 + 2A_1x_1^2 = \frac{2}{3}$, i.e. $A_0x_0^2 + A_1x_1^2 = \frac{1}{3}$.

With $x_0^2 = \frac{15 - 2\sqrt{30}}{35}$ and $x_1^2 = \frac{15 + 2\sqrt{30}}{35}$:

$$\begin{aligned} A_0 \cdot \frac{15 - 2\sqrt{30}}{35} + (1 - A_0) \cdot \frac{15 + 2\sqrt{30}}{35} &= \frac{1}{3} \\ \frac{15 + 2\sqrt{30}}{35} + A_0 \cdot \frac{-4\sqrt{30}}{35} &= \frac{1}{3} \end{aligned}$$

Thus

$$\begin{aligned} A_0 &= \frac{\frac{15 + 2\sqrt{30}}{35} - \frac{1}{3}}{\frac{4\sqrt{30}}{35}} = \frac{3(15 + 2\sqrt{30}) - 35}{4\sqrt{30}} = \frac{45 + 6\sqrt{30} - 35}{105} \cdot \frac{35}{4\sqrt{30}} = \frac{10 + 6\sqrt{30}}{12\sqrt{30}} \\ A_0 &= \frac{10 + 6\sqrt{30}}{12\sqrt{30}} = \frac{5 + 3\sqrt{30}}{6\sqrt{30}} = \frac{(5 + 3\sqrt{30})\sqrt{30}}{180} = \frac{5\sqrt{30} + 90}{180} = \frac{18 - \sqrt{30}}{36} \end{aligned}$$

Thus $A_1 = 1 - A_0$ which yields

$$A_1 = 1 - \frac{18 - \sqrt{30}}{36} = \frac{36}{36} - \frac{18 - \sqrt{30}}{36} = \frac{18 + \sqrt{30}}{36}$$

c) The Legendre polynomial $P_4(x)$ is proportional to $q(x)$ with the normalisation $P_4(1) = 1$.

$$\text{We have } q(1) = 1 - \frac{6}{7} + \frac{3}{35} = \frac{35 - 30 + 3}{35} = \frac{8}{35}.$$

Therefore:

$$\begin{aligned} P_4(x) &= \frac{q(x)}{q(1)} = \frac{35}{8} \left(x^4 - \frac{6}{7}x^2 + \frac{3}{35} \right) \\ &= \frac{1}{8} (35x^4 - 30x^2 + 3) \end{aligned}$$

d) Deriving the Legendre polynomial. Write $q(x) = x^2 + bx + c$. By symmetry $b = 0$, so $q(x) = x^2 + c$.

Orthogonality to 1:

$$\int_{-1}^1 (x^2 + c) dx = \frac{2}{3} + 2c = 0 \Rightarrow c = -\frac{1}{3}.$$

$$q(x) = x^2 - \frac{1}{3}$$

Gauss nodes:

$$x^2 = \frac{1}{3} \Rightarrow x_{0,1} = \pm \frac{1}{\sqrt{3}}$$

Applying to the integral. We approximate:

$$I = \int_{-3\pi/\sqrt{3}}^{3\pi/\sqrt{3}} \sin^2(x) \cos(x) dx.$$

Note $-\frac{3\pi}{\sqrt{3}} = -\pi\sqrt{3}$ and $\frac{3\pi}{\sqrt{3}} = \pi\sqrt{3}$.

Change of variable so that the integral is evaluated between $[-1,1]$. from $[a, b] = [-\pi\sqrt{3}, \pi\sqrt{3}]$ to $[-1, 1]$ using the map

$$x = \frac{b-a}{2}t + \frac{b+a}{2} = \pi\sqrt{3}t, \quad dx = \pi\sqrt{3} dt.$$

So:

$$I = \pi\sqrt{3} \int_{-1}^1 \sin^2(\pi\sqrt{3}t) \cos(\pi\sqrt{3}t) dt.$$

2-point Gauss rule ($A_0 = A_1 = 1$, nodes $t = \pm \frac{1}{\sqrt{3}}$):

$$I \approx \pi\sqrt{3} \left[f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right) \right], \quad f(t) = \sin^2(\pi\sqrt{3}t) \cos(\pi\sqrt{3}t).$$

Evaluate at $t = \frac{1}{\sqrt{3}}$: $\pi\sqrt{3} \cdot \frac{1}{\sqrt{3}} = \pi$.

$$f\left(\frac{1}{\sqrt{3}}\right) = \sin^2(\pi) \cos(\pi) = 0.$$

Evaluate at $t = -\frac{1}{\sqrt{3}}$: $\pi\sqrt{3} \cdot \left(-\frac{1}{\sqrt{3}}\right) = -\pi$.

$$f\left(-\frac{1}{\sqrt{3}}\right) = \sin^2(-\pi) \cos(-\pi) = 0.$$

$$I \approx \pi\sqrt{3}(0 + 0) = 0.$$

Exact check. Using $\sin^2(x) \cos(x) = \frac{d}{dx} \left[\frac{\sin^3(x)}{3} \right]$:

$$I = \left[\frac{\sin^3(x)}{3} \right]_{-\pi\sqrt{3}}^{\pi\sqrt{3}} = \frac{\sin^3(\pi\sqrt{3})}{3} - \frac{\sin^3(-\pi\sqrt{3})}{3} = \frac{2 \sin^3(\pi\sqrt{3})}{3}.$$

Numerically $\pi\sqrt{3} \approx 5.441$, $\sin(5.441) \approx -0.7485$, so $I \approx \frac{2(-0.7485)^3}{3} \approx -0.2797$.

Note: The 2-point Gauss rule gives $I = 0$ because the integrand evaluates to zero exactly at the nodes $\pm\pi$; a higher-order rule is needed to capture the true value.

$$I_{\text{Gauss}} = 0, \quad I_{\text{exact}} = \frac{2 \sin^3(\pi\sqrt{3})}{3} \approx -0.2797$$

Exercise 2: For the function $f(x) = (x + 1) \sin\left(\frac{\pi}{2}(x + 1)\right)$, evaluate the integral

$$\int_0^1 f(x) dx$$

using four subintervals for

- Trapezium rule,
- Simpson's rule and
- Compute the error in each case against the exact solution.

Four subintervals means five points, with a step size $h = \frac{1-0}{4} = 0.25$. the

i	x_i	$f(x_i)$
0	0	$\sin(\pi/2) = 1.0000$
1	1/4	$\frac{5}{4} \sin(5\pi/8) = 1.25 \times 0.9239 = 1.1548$
2	1/2	$\frac{3}{2} \sin(3\pi/4) = 1.50 \times 0.7071 = 1.0607$
3	3/4	$\frac{7}{4} \sin(7\pi/8) = 1.75 \times 0.3827 = 0.6697$
4	1	$2 \sin(\pi) = 0$

a) For the Trapezium rule:

$$\begin{aligned}
 T &= \frac{h}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)] \\
 &= \frac{0.25}{2} [1.0000 + 2(1.1548) + 2(1.0607) + 2(0.6697) + 0] \\
 &= 0.125 [1.0000 + 2.3096 + 2.1214 + 1.3394] \\
 &= 0.125 \times 6.7704 \\
 &\approx 0.8463
 \end{aligned}$$

b) For Simpson's rule:

$$\begin{aligned}
 S &= \frac{h}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)] \\
 &= \frac{0.25}{3} [1.0000 + 4(1.1548) + 2(1.0607) + 4(0.6697) + 0] \\
 &= \frac{0.25}{3} [1.0000 + 4.6192 + 2.1214 + 2.6788] \\
 &= \frac{0.25}{3} \times 10.4194 \\
 &\approx 0.8683
 \end{aligned}$$

c) To Use integration by parts twice with $u = x + 1$, $dv = \sin\left(\frac{\pi}{2}(x + 1)\right) dx$:

$$\begin{aligned}
 \int (x + 1) \sin\left(\frac{\pi}{2}(x + 1)\right) dx &= -(x + 1) \cdot \frac{2}{\pi} \cos\left(\frac{\pi}{2}(x + 1)\right) + \frac{2}{\pi} \int \cos\left(\frac{\pi}{2}(x + 1)\right) dx \\
 &= -\frac{2(x + 1)}{\pi} \cos\left(\frac{\pi}{2}(x + 1)\right) + \frac{4}{\pi^2} \sin\left(\frac{\pi}{2}(x + 1)\right) + C.
 \end{aligned}$$

Evaluating on the interval $[0, 1]$ requires evaluating

$$\text{At } x = 1: \quad -\frac{4}{\pi} \cos(\pi) + \frac{4}{\pi^2} \sin(\pi) = \frac{4}{\pi}.$$

At $x = 0$: $-\frac{2}{\pi} \cos(\pi/2) + \frac{4}{\pi^2} \sin(\pi/2) = \frac{4}{\pi^2}$.

Hence

$$I_{\text{exact}} = \frac{4}{\pi} - \frac{4}{\pi^2} = \frac{4(\pi - 1)}{\pi^2} \approx \frac{4 \times 2.1416}{9.8696} \approx 0.8677.$$

Thus the errors are given by

Method	Result	Error $ I - I_{\text{exact}} $
Trapezium	0.8463	0.0214
Simpson	0.8683	0.0006

Simpson's rule is far more accurate, as expected from its $\mathcal{O}(h^4)$ error versus $\mathcal{O}(h^2)$ for the Trapezium rule.

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Exercise 3:

a) Show, by applying the Romberg algorithm to the Trapezium method, that the next improvement is Simpson's Rule.

b) For

$$\int_1^2 \frac{1}{x^3} dx = \frac{3}{8}.$$

Compute for four intervals, the trapezium, Simpson and the next refinement of the Romberg algorithm.

a) Let $T(h)$ denote the trapezium approximation with step h . The Euler–Maclaurin expansion gives an expression for the error as a power series in h

$$T(h) = I + c_1 h^2 + c_2 h^4 + \dots$$

Thus the error as a power series in $\frac{h}{2}$ is given as

$$T\left(\frac{h}{2}\right) = I + c_1 \frac{h^2}{4} + c_2 \frac{h^4}{16} + \dots$$

Eliminate the h^2 term by multiplying $T(h/2)$ by 4 and subtracting $T(h)$,

$$4T\left(\frac{h}{2}\right) - T(h) = 3I + \mathcal{O}(h^4) \frac{4T(h/2) - T(h)}{3} = I + \mathcal{O}(h^4).$$

Now recall Simpson's rule with stepsize h (i.e. n intervals of size h):

$$S(h) = \frac{h}{3} [f_0 + 4f_1 + 2f_2 + \dots + f_n].$$

One can show directly that $T(h/2) = \frac{1}{2}T(h) + \frac{h}{2} \sum f_{\text{mid}}$, and that

$$\frac{4T(h/2) - T(h)}{3} \equiv S(h).$$

Therefore, the first Romberg improvement of the Trapezium rule is exactly Simpson's rule. □

b) Compute Trapezium rule, then Simpson's rule, then apply the Romberg method on Simpson's rule. As $a = 1$, $b = 2$, with $n = 4$ subintervals, $h = 0.25$, $x_i = 1, 5/4, 3/2, 7/4$ and 2 .

x_i	$f(x_i) = x_i^{-3}$
1.00	1.000000
1.25	$\frac{1}{(5/4)^3} = \frac{1}{5^3/4^3} = \frac{4^3}{5^3} = \frac{64}{125} = 0.512$
1.50	$\frac{8}{27} = 0.296296$
1.75	$\frac{27}{64} = 0.186589$
2.00	$\frac{1}{8} = 0.125000$

Trapezium rule ($n = 4$, $h = 0.25$):

$$\begin{aligned} T_1 &= \frac{0.25}{2} [1 + 2(0.512) + 2(0.296296) + 2(0.186589) + 0.125] \\ &= 0.125 [1 + 1.024 + 0.592593 + 0.373178 + 0.125] \\ &= 0.125 \times 3.114771 \\ &\approx 0.389346 \end{aligned}$$

We also need T_0 , the trapezium rule with $n = 2$ ($h = 0.5$), for the Romberg step:

$$\begin{aligned} T_0 &= \frac{0.5}{2} [f(1) + 2f(1.5) + f(2)] \\ &= 0.25 [1 + 0.592593 + 0.125] \\ &= 0.25 \times 1.717593 \\ &= 0.429398 \end{aligned}$$

Simpson's rule (i.e. the first Romberg step):

$$\begin{aligned} R_{2,1} &= \frac{4T_1 - T_0}{3} = \frac{4(0.389332) - 0.429398}{3} \\ &= \frac{1.557329 - 0.429398}{3} \\ &= \frac{1.127931}{3} \\ &\approx 0.375977 \end{aligned}$$

For the next Romberg step, we also need S_0 (Simpson with $n = 2$, i.e. $h = 0.5$):

$$R_{1,1} = \frac{4T_0 - T_{-1}}{3},$$

where T_{-1} is the trapezium with $n = 1$ ($h = 1$):

$$T_{-1} = \frac{1}{2} [f(1) + f(2)] = \frac{1}{2} (1 + 0.125) = 0.5625$$

thus

$$R_{1,1} = \frac{4(0.429398) - 0.5625}{3} = \frac{1.717593 - 0.5625}{3} = \frac{1.155093}{3} = 0.385031.$$

Next Romberg refinement (eliminates h^4 error):

$$\begin{aligned} R_{2,2} &= \frac{16R_{2,1} - R_{1,1}}{15} \\ &= \frac{16(0.375977) - 0.385031}{15} \\ &= \frac{6.015632 - 0.385031}{15} \\ &= \frac{5.630601}{15} \\ &\approx 0.375373 \end{aligned}$$

Method	Value	Error $ I - \text{value} $
Trapezium ($n = 4$)	0.389332	0.014332
Simpson ($n = 4$)	0.375977	0.000977
Romberg $R_{2,2}$	0.375373	0.000373
Exact	0.375000	—

Each Romberg level increases the order of accuracy by two powers of h .