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## CTMS-MAT-13: Numerical Methods

Problem Sheet 6 Solutions. Released 5 May 2026

**Exercise 1:** Consider the linear ordinary differential equation

$$y''(t) = -\alpha y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

- a) Convert this second-order ordinary differential equation into a system of two coupled first-order ODEs, one in  $y(t)$  and one in  $y'(t)$ . Write the system as a vector-valued ordinary differential equation for  $\vec{v}(t) = (y(t), y'(t))^T$ , in the form  $f(\vec{v}) = A\vec{v}$ .

- b) Show that the backward Euler method can be written as

$$\vec{u}_{n+1} = (I - hA)^{-1} \vec{u}_n$$

and provide the full system for  $\vec{u}_{n+1}$  for the ODE presented above.

- c) Noting that  $f_n = A\vec{u}_n$ , so that  $f(\vec{v}_n + hf_n) = A\vec{v}_n + hA^2\vec{v}_n$ , show that Heun's method for the case of the vector-valued ODE given above is

$$\vec{v}_{n+1}(t) = \begin{pmatrix} 1 + \frac{h^2}{2} & h\left(1 - \alpha\frac{h}{2}\right) \\ h\left(1 - \alpha\frac{h}{2}\right) & 1 - h\left(\alpha - \frac{h}{2}(1 + \alpha^2)\right) \end{pmatrix} \vec{v}_n(t).$$

- d) With  $\alpha = 2.5$ , calculate  $y(0.3)$  and  $y'(0.3)$  using the backward Euler method with  $h = 0.1$ .  
 e) With  $\alpha = 2.5$ , calculate  $y(0.3)$  and  $y'(0.3)$  using the Crank-Nicolson method with  $h = 0.1$ .

- a) Let  $v_1 = y(t)$  and  $v_2 = y'(t)$ . Then the differential equations are

$$\begin{aligned} v_1' &= v_2, \\ v_2' &= -\alpha v_2 + v_1. \end{aligned}$$

In matrix form this is the linear equation  $\vec{v}' = A\vec{v}$ , explicitly

$$\frac{d}{dt} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix},$$

where  $A = \begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix}$  with initial condition  $\vec{v}(0) = (1, 1)^T$ . □

- b) The backward Euler method evaluates the right-hand side at the new time level:

$$\begin{aligned} \vec{u}_{n+1} &= \vec{u}_n + hA\vec{u}_{n+1} \\ \vec{u}_{n+1} - hA\vec{u}_{n+1} &= \vec{u}_n \\ (I - hA)\vec{u}_{n+1} &= \vec{u}_n \\ \vec{u}_{n+1} &= (I - hA)^{-1} \vec{u}_n \end{aligned}$$

For the given ordinary differential equation:

$$I - hA = \begin{pmatrix} 1 & -h \\ -h & 1 + h\alpha \end{pmatrix}$$

The determinant is  $\det(I - hA) = 1 + h\alpha - h^2$ , so the inverse is:

$$(I - hA)^{-1} = \frac{1}{1 + h\alpha - h^2} \begin{pmatrix} 1 + h\alpha & h \\ h & 1 \end{pmatrix}$$

The full system is therefore:

$$\vec{u}_{n+1} = \frac{1}{1 + h\alpha - h^2} \begin{pmatrix} 1 + h\alpha & h \\ h & 1 \end{pmatrix} \vec{u}_n$$

c) Heun's method is:

$$\vec{v}_{n+1} = \vec{v}_n + \frac{h}{2} (f(\vec{v}_n) + f(\vec{v}_n + hf(\vec{v}_n)))$$

With  $f(\vec{v}) = A\vec{v}$ :

$$\begin{aligned} f(\vec{v}_n) &= A\vec{v}_n \\ f(\vec{v}_n + hA\vec{v}_n) &= A(I + hA)\vec{v}_n = (A + hA^2)\vec{v}_n \end{aligned}$$

Thus:

$$\vec{v}_{n+1} = \left( I + hA + \frac{h^2}{2} A^2 \right) \vec{v}_n$$

Computing  $A^2$ :

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & -\alpha \end{pmatrix}^2 = \begin{pmatrix} 1 & -\alpha \\ -\alpha & 1 + \alpha^2 \end{pmatrix}$$

Then:

$$I + hA + \frac{h^2}{2} A^2 = \begin{pmatrix} 1 + \frac{h^2}{2} & h\left(1 - \alpha\frac{h}{2}\right) \\ h\left(1 - \alpha\frac{h}{2}\right) & 1 - h\left(\alpha - \frac{h}{2}(1 + \alpha^2)\right) \end{pmatrix}$$

d) With  $\alpha = 2.5$  and  $h = 0.1$ , we have  $1 + h\alpha - h^2 = 1.24$ , so:

$$\Rightarrow \frac{1}{1 + h\alpha - h^2} \begin{pmatrix} 1 + h\alpha & h \\ h & 1 \end{pmatrix} = \frac{1}{1.24} \begin{pmatrix} 1.25 & 0.1 \\ 0.1 & 1 \end{pmatrix}$$

Starting from  $\vec{u}_0 = (1, 1)^T$ , three steps give:

$$\vec{u}_1 = \frac{1}{1.24} \begin{pmatrix} 1.25 & 0.1 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{1.24} \begin{pmatrix} 1.35 \\ 1.10 \end{pmatrix} = \begin{pmatrix} 1.08871 \\ 0.88710 \end{pmatrix}$$

$$\vec{u}_2 = \frac{1}{1.24} \begin{pmatrix} 1.25 & 0.1 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} 1.08871 \\ 0.88710 \end{pmatrix} = \frac{1}{1.24} \begin{pmatrix} 1.44960 \\ 0.99581 \end{pmatrix} = \begin{pmatrix} 1.16903 \\ 0.80307 \end{pmatrix}$$

$$\vec{u}_3 = \frac{1}{1.24} \begin{pmatrix} 1.25 & 0.1 \\ 0.1 & 1 \end{pmatrix} \begin{pmatrix} 1.16903 \\ 0.80307 \end{pmatrix} = \frac{1}{1.24} \begin{pmatrix} 1.54160 \\ 0.91997 \end{pmatrix} = \begin{pmatrix} 1.24323 \\ 0.74191 \end{pmatrix}$$

Therefore  $y(0.3) \approx 1.24323$  and  $y'(0.3) \approx 0.74191$ .

e) The Crank-Nicolson method gives  $\tilde{u}_{n+1} = M\tilde{u}_n$  with:

$$M = \left(I - \frac{h}{2}A\right)^{-1} \left(I + \frac{h}{2}A\right)$$

With  $h = 0.1$  and  $\alpha = 2.5$ :

$$I - \frac{h}{2}A = \begin{pmatrix} 1 & -0.05 \\ -0.05 & 1.125 \end{pmatrix}$$

$$I + \frac{h}{2}A = \begin{pmatrix} 1 & 0.05 \\ 0.05 & 0.875 \end{pmatrix}$$

The determinant is  $\det\left(I - \frac{h}{2}A\right) = 1 + h\alpha/2 - h^2/4 = 1.1225$ , and:

$$\begin{aligned} I - \frac{h}{2}A &= \begin{pmatrix} 1 & \frac{h}{2} \\ \frac{h}{2} & 1 + \frac{h}{2}\alpha \end{pmatrix} \\ \Rightarrow \left(I - \frac{h}{2}A\right)^{-1} &= \frac{1}{1 + h\alpha/2 - h^2/4} \begin{pmatrix} 1 + h\alpha/2 & h/2 \\ h/2 & 1 \end{pmatrix} \end{aligned}$$

So the linear equation for the Crank-Nicolson scheme is:

$$M = \frac{1}{1.1225} \begin{pmatrix} 1.1275 & 0.09938 \\ 0.09938 & 0.87750 \end{pmatrix}$$

Starting from  $\tilde{u}_0 = (1, 1)^T$ :

$$\begin{aligned} \tilde{u}_1 &= M\tilde{u}_0 \\ &= \frac{1}{1.1225} \begin{pmatrix} 1.1275 & 0.09938 \\ 0.09938 & 0.87750 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{1.1225} \begin{pmatrix} 1.22688 \\ 0.97688 \end{pmatrix} = \begin{pmatrix} 1.09309 \\ 0.87025 \end{pmatrix} \end{aligned}$$

$$\tilde{u}_2 = \frac{1}{1.1225} \begin{pmatrix} 1.1275 & 0.09938 \\ 0.09938 & 0.87750 \end{pmatrix} \begin{pmatrix} 1.09309 \\ 0.87025 \end{pmatrix} = \frac{1}{1.1225} \begin{pmatrix} 1.32108 \\ 0.87182 \end{pmatrix} = \begin{pmatrix} 1.17699 \\ 0.77666 \end{pmatrix}$$

$$\tilde{u}_3 = \frac{1}{1.1225} \begin{pmatrix} 1.1275 & 0.09938 \\ 0.09938 & 0.87750 \end{pmatrix} \begin{pmatrix} 1.17699 \\ 0.77666 \end{pmatrix} = \begin{pmatrix} 1.25270 \\ 0.75614 \end{pmatrix}$$

Therefore  $y(0.3) \approx 1.25270$  and  $y'(0.3) \approx 0.75614$ .

**Exercise 2:** Consider the ordinary differential equation

$$y'(t) = -y(t) + \ln(t+1) \quad \text{with} \quad y(0) = 1.$$

Showing all working, calculate four time steps of the approximate solution using the fourth-order Runge-Kutta method with step size  $h = 0.1$ .

At each step the method computes

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ k_2 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right), \\ k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2}k_2\right), \\ k_4 &= f(t_n + h, y_n + hk_3), \\ y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4). \end{aligned}$$

*Step 1.*  $t_0 = 0 \rightarrow t_1 = 0.1, \quad y_0 = 1.$

$$\begin{aligned} k_1 &= f(0, 1) = -1 + \ln(1) = -1 + 0 = -1.000000, \\ k_2 &= f(0.05, 1 + 0.05(-1)) = f(0.05, 0.95) = -0.95 + \ln(1.05) \\ &= -0.95 + 0.048790 = -0.901210, \\ k_3 &= f(0.05, 1 + 0.05(-0.901210)) = f(0.05, 0.954939) = -0.954939 + \ln(1.05) \\ &= -0.954939 + 0.048790 = -0.906149, \\ k_4 &= f(0.1, 1 + 0.1(-0.906149)) = f(0.1, 0.909385) = -0.909385 + \ln(1.1) \\ &= -0.909385 + 0.095310 = -0.814075, \\ y_1 &= 1 + \frac{0.1}{6}(-1.000000 + 2(-0.901210) + 2(-0.906149) + (-0.814075)) \\ &= 1 + \frac{0.1}{6}(-1.000000 - 1.802420 - 1.812298 - 0.814075) \\ &= 1 + \frac{0.1}{6}(-5.428793) \\ &= 1 - 0.090480 \\ &= 0.909520. \end{aligned}$$

*Step 2.*  $t_1 = 0.1 \rightarrow t_2 = 0.2, \quad y_1 = 0.909520.$

$$\begin{aligned} k_1 &= f(0.1, 0.909520) = -0.909520 + \ln(1.1) = -0.909520 + 0.095310 = -0.814210, \\ k_2 &= f(0.15, 0.909520 + 0.05(-0.814210)) = f(0.15, 0.868809) \\ &= -0.868809 + \ln(1.15) = -0.868809 + 0.139762 = -0.729047, \\ k_3 &= f(0.15, 0.909520 + 0.05(-0.729047)) = f(0.15, 0.873068) \\ &= -0.873068 + \ln(1.15) = -0.873068 + 0.139762 = -0.733306, \\ k_4 &= f(0.2, 0.909520 + 0.1(-0.733306)) = f(0.2, 0.836189) \\ &= -0.836189 + \ln(1.2) = -0.836189 + 0.182322 = -0.653867, \\ y_2 &= 0.909520 + \frac{0.1}{6}(-0.814210 + 2(-0.729047) + 2(-0.733306) + (-0.653867)) \\ &= 0.909520 + \frac{0.1}{6}(-0.814210 - 1.458094 - 1.466612 - 0.653867) \\ &= 0.909520 + \frac{0.1}{6}(-4.392783) \\ &= 0.909520 - 0.073213 \\ &= 0.836307. \end{aligned}$$

Step 3.  $t_2 = 0.2 \rightarrow t_3 = 0.3$ ,  $y_2 = 0.836307$ .

$$k_1 = f(0.2, 0.836307) = -0.836307 + \ln(1.2) = -0.836307 + 0.182322 = -0.653985,$$

$$k_2 = f(0.25, 0.836307 + 0.05(-0.653985)) = f(0.25, 0.803608) \\ = -0.803608 + \ln(1.25) = -0.803608 + 0.223144 = -0.580464,$$

$$k_3 = f(0.25, 0.836307 + 0.05(-0.580464)) = f(0.25, 0.807284) \\ = -0.807284 + \ln(1.25) = -0.807284 + 0.223144 = -0.584140,$$

$$k_4 = f(0.3, 0.836307 + 0.1(-0.584140)) = f(0.3, 0.777893) \\ = -0.777893 + \ln(1.3) = -0.777893 + 0.262364 = -0.515529,$$

$$y_3 = 0.836307 + \frac{0.1}{6}(-0.653985 + 2(-0.580464) + 2(-0.584140) + (-0.515529)) \\ = 0.836307 + \frac{0.1}{6}(-0.653985 - 1.160928 - 1.168280 - 0.515529) \\ = 0.836307 + \frac{0.1}{6}(-3.498722) \\ = 0.836307 - 0.058312 \\ = 0.777995.$$

Step 4.  $t_3 = 0.3 \rightarrow t_4 = 0.4$ ,  $y_3 = 0.777995$ .

$$k_1 = f(0.3, 0.777995) = -0.777995 + \ln(1.3) = -0.777995 + 0.262364 = -0.515631,$$

$$k_2 = f(0.35, 0.777995 + 0.05(-0.515631)) = f(0.35, 0.752213) \\ = -0.752213 + \ln(1.35) = -0.752213 + 0.300105 = -0.452108,$$

$$k_3 = f(0.35, 0.777995 + 0.05(-0.452108)) = f(0.35, 0.755390) \\ = -0.755390 + \ln(1.35) = -0.755390 + 0.300105 = -0.455285,$$

$$k_4 = f(0.4, 0.777995 + 0.1(-0.455285)) = f(0.4, 0.732467) \\ = -0.732467 + \ln(1.4) = -0.732467 + 0.336472 = -0.395995,$$

$$y_4 = 0.777995 + \frac{0.1}{6}(-0.515631 + 2(-0.452108) + 2(-0.455285) + (-0.395995)) \\ = 0.777995 + \frac{0.1}{6}(-0.515631 - 0.904216 - 0.910570 - 0.395995) \\ = 0.777995 + \frac{0.1}{6}(-2.726412) \\ = 0.777995 - 0.045440 \\ = 0.732555.$$

The results are summarised in the following table.

$n$	$t_n$	$k_1$	$k_2$	$k_3$	$k_4$	$y_n$
0	0.0	-1.000000	-0.901210	-0.906149	-0.814075	1.000000
1	0.1	-0.814210	-0.729047	-0.733306	-0.653867	0.909520
2	0.2	-0.653985	-0.580464	-0.584140	-0.515529	0.836307
3	0.3	-0.515631	-0.452108	-0.455285	-0.395995	0.777995
4						0.732555

**Exercise 3:** With  $h = \frac{1}{4}$  compute  $y(1)$  using Heun's method for the ordinary differential equation given by

$$ty'(t) = y(t) \sin(t) \quad \text{with} \quad y(0) = 2.$$

Rewrite the ODE as

$$y' = f(t, y) = \frac{y \sin(t)}{t}, \quad t > 0.$$

Note that at  $t = 0$  we use the limit  $\lim_{t \rightarrow 0} \frac{\sin(t)}{t} = 1$ , so

$$f(0, y) = y.$$

Thus initial condition is well-defined, with  $y_0 = y(0) = 2$ . For  $h = \frac{1}{4}$  from  $t = 0$  to  $t = 1$ , gives four steps at  $t_0 = 0$ ,  $t_1 = \frac{1}{4}$ ,  $t_2 = \frac{1}{2}$ ,  $t_3 = \frac{3}{4}$ ,  $t_4 = 1$ . At each step Heun's method proceeds as follows:

$$\begin{aligned} k_1 &= f(t_n, y_n), \\ \tilde{y}_{n+1} &= y_n + h k_1 \\ k_2 &= f(t_{n+1}, \tilde{y}_{n+1}), \\ y_{n+1} &= y_n + \frac{h}{2}(k_1 + k_2). \end{aligned}$$

Step 1:  $t_0 = 0 \rightarrow t_1 = \frac{1}{4}$

$$\begin{aligned} k_1 &= f(0, 2) = 2, \\ \Rightarrow \tilde{y}_1 &= 2 + \frac{1}{4}(2) = 2.5, \\ k_2 &= f\left(\frac{1}{4}, 2.5\right) = \frac{2.5 \sin(0.25)}{0.25} = \frac{2.5 \times 0.247404}{0.25} = 2.474040, \\ \Rightarrow y_1 &= 2 + \frac{1/4}{2}(2 + 2.474040) = 2 + \frac{1}{8}(4.474040) = 2 + 0.559255 = 2.559255. \end{aligned}$$

Step 2:  $t_1 = \frac{1}{4} \rightarrow t_2 = \frac{1}{2}$

$$\begin{aligned} k_1 &= f\left(\frac{1}{4}, 2.559255\right) = \frac{2.559255 \sin(0.25)}{0.25} = \frac{2.559255 \times 0.247404}{0.25} = 2.530816, \\ \tilde{y}_2 &= 2.559255 + \frac{1}{4}(2.530816) = 2.559255 + 0.632704 = 3.191959, \\ k_2 &= f\left(\frac{1}{2}, 3.191959\right) = \frac{3.191959 \sin(0.5)}{0.5} = \frac{3.191959 \times 0.479426}{0.5} = 3.061963, \\ y_2 &= 2.559255 + \frac{1}{8}(2.530816 + 3.061963) = 2.559255 + \frac{1}{8}(5.592779) = 2.559255 + 0.699097 = 3.258352. \end{aligned}$$

Step 3:  $t_2 = \frac{1}{2} \rightarrow t_3 = \frac{3}{4}$

$$\begin{aligned} k_1 &= f\left(\frac{1}{2}, 3.258352\right) = \frac{3.258352 \sin(0.5)}{0.5} = \frac{3.258352 \times 0.479426}{0.5} = 3.125628, \\ \tilde{y}_3 &= 3.258352 + \frac{1}{4}(3.125628) = 3.258352 + 0.781407 = 4.039759, \\ k_2 &= f\left(\frac{3}{4}, 4.039759\right) = \frac{4.039759 \sin(0.75)}{0.75} = \frac{4.039759 \times 0.681639}{0.75} = 3.671726, \\ y_3 &= 3.258352 + \frac{1}{8}(3.125628 + 3.671726) = 3.258352 + \frac{1}{8}(6.797354) = 3.258352 + 0.849669 = 4.108021. \end{aligned}$$

Step 4:  $t_3 = \frac{3}{4} \rightarrow t_4 = 1$

$$k_1 = f\left(\frac{3}{4}, 4.108021\right) = \frac{4.108021 \sin(0.75)}{0.75} = \frac{4.108021 \times 0.681639}{0.75} = 3.733783,$$

$$\Rightarrow \tilde{y}_4 = 4.108021 + \frac{1}{4}(3.733783) = 4.108021 + 0.933446 = 5.041467,$$

$$k_2 = f(1, 5.041467) = \frac{5.041467 \sin(1)}{1} = 5.041467 \times 0.841471 = 4.242022,$$

$$\Rightarrow y_4 = 4.108021 + \frac{1}{8}(3.733783 + 4.242022) = 4.108021 + \frac{1}{8}(7.975805) = 4.108021 + 0.996976 = 5.104997.$$

Thus:

$n$	$t_n$	$k_1$	$k_2$	$y_n$
0	0.00	2.000000	2.474040	2.000000
1	0.25	2.530816	3.061963	2.559255
2	0.50	3.125628	3.671726	3.258352
3	0.75	3.733783	4.242022	4.108021
4	1.00	—	—	5.104997

The approximation using Heun's method is

$$y(1) = 5.104997.$$