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# JTMS-MAT-13: Numerical Methods

## Sample Multiple Choice Exam Solutions

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**Question 1**:
What is the solution after the first step of the Newton method for the function $f(x) = 3x^3 + 5x + 1$ with the initial guess $x_0 = 1$

- ○ $x_1 = 3/4$
- ○ $x_1 = 2/14$
- ○ $x_1 = 1/4$
- ○ $x_1 = 9/10$
- ✗ $x_1 = 5/14$

The derivative $f'(x)$ of $f(x) = 3x^3 + 5x + 1$ is:

$$f'(x) = 9x^2 + 5.$$

Evaluate $f(x)$ and $f'(x)$ at $x_0 = 1$:

$$f(1) = 3(1)^3 + 5(1) + 1 = 3 + 5 + 1 = 9 \quad \text{and} \quad f'(1) = 9(1)^2 + 5 = 9 + 5 = 14.$$

Apply the Newton method formula to get the next approximation, $x_1$:

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= 1 - \frac{9}{14}, \\ &= \frac{14 - 9}{14} = \frac{5}{14}. \end{aligned}$$

**Question 2**:
What conditions need to hold for the bisection method to converge to a root of the function $f(x)$ within the interval $[a,b]$?

- ○ $f \in \mathcal{C}^2([a,b])$
- ○ $f' \neq 0$
- ○ $f \in \mathcal{C}^1([a,b])$
- ○ $f(a)f(b) < 0$
- ☒ $f \in \mathcal{C}([a,b])$
- ☒ $f(a) < f(b)$

For the bisection method to converge to a root of the function $f(x)$ within the interval $[a,b]$, the following conditions must be met:

- **Continuity**: The function $f(x)$ must be continuous on the closed interval $[a,b]$, i.e. $f \in \mathcal{C}([a,b])$
- **Opposite Signs at Endpoints**: The function values at the endpoints of the interval must have opposite signs, i.e., $f(a) \cdot f(b) < 0$. This ensures that there is at least one root of the function in the interval $[a,b]$ by the Intermediate Value Theorem.

If both conditions are satisfied, the bisection method will converge to a root of $f(x)$ within the interval $[a,b]$.

**Question 3**:
What are the correct values for the coefficients $a$ and $b$ for the least squares approximation for $h(x) = a + bx^2$ and the data pairs $(-1,1)$, $(0,2)$ and $(1,1)$?

- ○ $a = 1$
- ○ $b = 2$
- ☒ $b = -1$
- ○ $b = 0$
- ○ $a = -1$
- ☒ $a = 2$
- ○ $a = -1$
- ○ $b = -2$
- ○ $b = 1$
- ○ $a = 0$
- ○ $a = 1$
- ○ $a = -2$

We want to minimize the sum of squared differences

$$S = \sum_{i=1}^{n} [y_i - h(x_i)]^2, \quad \text{where} \quad h(x) = a + bx^2.$$

There are three data points, but only two parameters to fit the data. Note that the data is symmetric, so the coefficents of any odd terms in fitting function would be zero. From these points, set up equations based on the model:

$$1 = a + b(-1)^2,$$
$$2 = a + b(0)^2,$$
$$1 = a + b(1)^2$$

Thus, the coefficients that best fit the data in a least squares sense are: - $a = 2$ - $b = -1$ The least squares approximation polynomial is:

$$h(x) = 2 - x^2$$

**Question 4**:
For the following Runge-Kutta method:

$$u_{n+1} = u_n + (h/6)\,(K_1 + 4K_2 + K_3)$$
$$\text{with} \quad K_1 = f(u_n, t_n), \quad K_2 = f(u_n + hK_1/2, t_n + h/2), \quad \text{and} \quad K_3 = f(u_n - hK_1 + 2hK_2, t_n + h)$$

select the correct Butcher array

○ $$\begin{array}{c|ccc} 1 & & & \\ 4 & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & -1 & 2 & \end{array}$$

○ $$\begin{array}{c|ccc} & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & 1 & 4 & 1 \end{array}$$

○ $$\begin{array}{c|ccc} \frac{1}{6} & & & \\ \frac{2}{3} & \frac{1}{2} & & \\ \frac{1}{6} & -1 & 2 & \\ \hline & -1 & 2 & \end{array}$$

○ $$\begin{array}{c|ccc} & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

☒ $$\begin{array}{c|ccc} & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ 1 & -1 & 2 & \\ \hline & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \end{array}$$

Identifying first the weights $b$, yields $b = (1/6, 4/6, 1/6) = (1/6, 2/3, 1/6)$. Which, by looking at the bottom row of the Butcher arrays eliminates the first three cases.

The time increment vector $c$ is the same for both, but the matrix $A$ is different.

By looking at $K_3$, we see that the first argument has $u_n - hK_1 + 2hK_2$, which corresponds to the final choice.

**Question 5**:
For which of the following matrices can you perform a Cholesky decomposition?

- [x] $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 14 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 3 & 8 \end{pmatrix}$.
- [ ] $\begin{pmatrix} 2 & 1 & 7 \\ 2 & 2 & 3 \\ 2 & 2 & 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}$.
- [ ] $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

In order to perform Cholesky decomposition, the matrix must be symmetric(thus square) and positive definite. Only the first matrix is symmetric. All that remains to show is that it is positive definite. There are two ways to do this: either to show all eigenvalues are positive, or that for an arbitrary vector $z$, then $z^T Az > 0$.

Compute the determinant of $A - \lambda I$:

$$\begin{aligned}\det(A-\lambda I) &= \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 5-\lambda & 8 \\ 3 & 8 & 14-\lambda \end{vmatrix} \\ &= (1-\lambda)\begin{vmatrix} 5-\lambda & 8 \\ 8 & 14-\lambda \end{vmatrix} - 2\begin{vmatrix} 2 & 8 \\ 3 & 14-\lambda \end{vmatrix} + 3\begin{vmatrix} 2 & 5-\lambda \\ 3 & 8 \end{vmatrix}\end{aligned}$$

Simplify each of these 2×2 determinants:

$$\begin{vmatrix} 5-\lambda & 8 \\ 8 & 14-\lambda \end{vmatrix} = (5-\lambda)(14-\lambda) - 64 = \lambda^2 - 19\lambda + 6$$

$$\begin{vmatrix} 2 & 8 \\ 3 & 14-\lambda \end{vmatrix} = 2(14-\lambda) - 24 = 2\,(2-\lambda)$$

$$\begin{vmatrix} 2 & 5-\lambda \\ 3 & 8 \end{vmatrix} = 16 - 3(5-\lambda) = 1 + 3\lambda$$

Substituting these back:

$$\begin{aligned}\det(A-\lambda I) &= (1-\lambda)\left[\lambda^2 - 19\lambda + 6\right] - 4\left[2-\lambda\right] + 3[1+3\lambda] \\ &= -\lambda^3 + 20\lambda^2 - 12\lambda + 1\end{aligned}$$

It can be shown (numerically!) that this has three real positive roots: 19.38358088, 0.09987551, 0.51654362.

Alternatively,

$$\begin{aligned} z^T Az &= \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 14 \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \\ &= \begin{pmatrix} z_1 & z_2 & z_3 \end{pmatrix}\begin{pmatrix} 1z_1 + 2z_2 + 3z_3 \\ 2z_1 + 5z_2 + 8z_3 \\ 3z_1 + 8z_2 + 14z_3 \end{pmatrix} \\ &= z_1(1z_1 + 2z_2 + 3z_3) + z_2(2z_1 + 5z_2 + 8z_3) + z_3(3z_1 + 8z_2 + 14z_3) \\ &= z_1^2 + 4z_1z_2 + 6z_1z_3 + 5z_2^2 + 16z_2z_3 + 14z_3^2 \end{aligned}$$

Since $z_1, z_2$, and $z_3$ are all positive, each term in this expression is positive, thus the matrix is positive definite.

**Question 6**:
Which of the following systems is/are overdetermined?

- ○ $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 3 & 8 & 14 \end{pmatrix}$
- ☒ $\begin{pmatrix} 1 & 2 \\ 2 & 5 \\ 3 & 8 \end{pmatrix}$
- ○ $\begin{pmatrix} 2 & 1 & 7 \\ 2 & 2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$
- ☒ $\begin{pmatrix} 2 & 1 & 1 \\ 2 & 5 & 1 \end{pmatrix}$
- ☒ $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

**Question 7**:
Given the following data:

| $i$ | 0 | 1 | 2 |
|---|---|---|---|
| $x_i$ | 0 | 1 | 3 |
| $y_i$ | 1 | 3 | 2 |

Using polynomial interpolation, what is the value of $y(2)$?

- ○ $y(2) = 3/2$
- ○ $y(2) = 3/4$
- ☒ $y(2) = 10/3$
- ○ $y(2) = 11/4$
- ○ $y(2) = 4/9$
- ○ $y(2) = 0$

There is a choice of which polynomial basis can be used: Lagrange, Newton, etc. For Lagrange polynomials:

$$\begin{aligned}
L_0(x) &= \frac{(x-1)(x-3)}{(0-1)(0-3)} = \frac{(x-1)(x-3)}{3},\\
L_1(x) &= \frac{(x-0)(x-3)}{(1-0)(1-3)} = -\frac{x(x-3)}{2} = \frac{x(3-x)}{2},\\
L_2(x) &= \frac{(x-0)(x-1)}{(3-0)(3-1)} = \frac{x(x-1)}{6}.
\end{aligned}$$

The interpolating polynomial is

$$\begin{aligned}
p(x) &= 1\cdot\frac{(x-1)(x-3)}{3} + 3\cdot\frac{x(3-x)}{2} + 2\cdot\frac{x(x-1)}{6}\\
&= \frac{(x-1)(x-3)}{3} - \frac{3x(x-3)}{2} + \frac{x(x-1)}{3}\\
&= \frac{1}{3}(x^2-4x+3) - \frac{3}{2}(x^2-3x) + \frac{1}{3}(x^2-x)\\
&= \frac{1}{3}x^2 - \frac{4}{3}x + 1 - \frac{3}{2}x^2 + \frac{9}{2}x + \frac{1}{3}x^2 - \frac{1}{3}x\\
&= \left(\frac{1}{3} - \frac{3}{2} + \frac{1}{3}\right)x^2 + \left(-\frac{4}{3} + \frac{9}{2} - \frac{1}{3}\right)x + 1
\end{aligned}$$

Thus, the polynomial interpolant that passes through the points is given by

$$p(x) = -\frac{5}{6}x^2 + \frac{17}{6}x + 1.$$

At the value $x = 2$ this yields 10/3.

**Question 8**:
Using the trapezoidal rule with three subintervals, approximate the integral

$$T = \int_0^4 x^2 + 1 \, dx$$

and find the approximate solution as

- ○ $T = 29.000$
- ○ $T = 19.333$
- ○ $T = 30.100$
- ○ $T = 28.519$
- ○ $T = 25.650$
- ☒ $T = 26.519$

Calculate the width $h$ of each subinterval:

$$h = \frac{b-a}{n} = \frac{4-0}{3} = \frac{4}{3}.$$

Determine the $x$-values where the function will be evaluated:

$$x_0 = 0, \quad x_1 = \frac{4}{3}, \quad x_2 = \frac{8}{3}, \quad x_3 = 4.$$

Evaluate the function $f(x)$ at these $x$-values:

$$\begin{aligned}
f(x_0) &= f(0) = 0^2 + 1 = 1,\\
f(x_1) &= f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^2 + 1 = \frac{16}{9} + 1 = \frac{25}{9},\\
f(x_2) &= f\left(\frac{8}{3}\right) = \left(\frac{8}{3}\right)^2 + 1 = \frac{64}{9} + 1 = \frac{73}{9},\\
f(x_3) &= f(4) = 4^2 + 1 = 17
\end{aligned}$$

Apply the Trapezium rule:

$$\begin{aligned}
I &= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right]\\
&= \frac{4/3}{2}\left[1 + 2 \times \frac{25}{9} + 2 \times \frac{73}{9} + 17\right]\\
&= \frac{2}{3}\left[1 + \frac{50}{9} + \frac{146}{9} + 17\right]\\
&= \frac{2}{3}\left[\frac{9}{9} + \frac{50}{9} + \frac{146}{9} + \frac{153}{9}\right]\\
&= \frac{2}{3} \times \frac{358}{9} = \frac{716}{27} \approx 26.519
\end{aligned}$$

Therefore, the approximate value of the integral using the Trapezium rule with three subintervals is approximately 26.519.

**Question 9**:
Given the system of non-linear equations

$$f(x_1, x_2) = \begin{pmatrix} 2x_1 + \cos(x_2) \\ x_1^3 + x_1 \cos(x_2) \end{pmatrix}$$

what is the Jacobian matrix that needs to be inverted for the Newton method?

- [x] $\begin{pmatrix} 2 & -\sin(x_2) \\ 3x_1^2 + \cos(x_2) & -x_1 \sin(x_2) \end{pmatrix}$
- [ ] $\begin{pmatrix} -\sin(x_2) & -x_2 \sin(x_2) \\ 3x_1 & 2 \end{pmatrix}$
- [ ] $\begin{pmatrix} 2 & -\sin(x_2) \\ 3x_1^2 + \cos(x_2) & x_1 \cos(x_2) \end{pmatrix}$
- [ ] $\begin{pmatrix} 2 & 3x_1^2 \\ \cos(x_2) & -x_1 \sin(x_2) \end{pmatrix}$
- [ ] $\begin{pmatrix} 2x_1 & 2x_2 \\ \sin(x_1) & \cos(x_2) \end{pmatrix}$
- [ ] $\begin{pmatrix} -\sin(x_2) & 2 \\ -x_1 \sin(x_2) & 2x_1 \end{pmatrix}$

The Jacobian matrix $J$ is defined as:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix}$$

where $f_1(x_1, x_2) = 2x_1 + \cos(x_2)$ and $f_2(x_1, x_2) = x_1^3 + x_1 \cos(x_2)$.

Calculate $\frac{\partial f_1}{\partial x_1}$:

$$\frac{\partial}{\partial x_1}(2x_1 + \cos(x_2)) = 2$$

Calculate $\frac{\partial f_1}{\partial x_2}$:

$$\frac{\partial}{\partial x_2}(2x_1 + \cos(x_2)) = -\sin(x_2)$$

Calculate $\frac{\partial f_2}{\partial x_1}$:

$$\frac{\partial}{\partial x_1}(x_1^3 + x_1 \cos(x_2)) = 3x_1^2 + \cos(x_2)$$

Calculate $\frac{\partial f_2}{\partial x_2}$:

$$\frac{\partial}{\partial x_2}(x_1^3 + x_1 \cos(x_2)) = -x_1 \sin(x_2)$$

So, the Jacobian matrix $J$ for the function $f(x_1, x_2)$ is:

$$J = \begin{pmatrix} 2 & -\sin(x_2) \\ 3x_1^2 + \cos(x_2) & -x_1 \sin(x_2) \end{pmatrix}.$$

**Question 10**:
Which of the following statements is true?

- ○ The order of convergence of the bisection method is 1.
- ○ The optimal order of convergence of secant method is higher than that for Newton method.
- ○ Bisection method will always find a root if the function is continuous.
- ○ Newton method will always find a root if the derivative exists and is not equal to zero.
- ☒ For convergence in $[a, b]$, bisection method needs a continuous function on $[a, b]$, and $f(a)f(b) < 0$.
- ☒ Under certain conditions, Newton method has quadratic convergence.

The absolute error is halved at each step so the method converges linearly, with order half, not one.

The optimal order of convergence of the Secant method is approximately 1.618 – the golden ratio, $\varphi = (1 + \sqrt{5})/2$ – which is superlinear but less than quadratic. In contrast, Newton's method has a quadratic convergence rate, specifically with an order of 2.

The bisection method will find a root if the function is continuous in $[a, b]$, bisection method needs a continuous function on $[a, b]$, and $f(a)f(b) < 0$.

The initial guess for Newton's method must be sufficiently close to the root. Furthermore, it may be stuck in a cycle, depending on the second derivative.

**Question 11**:
Select those statements that are correct.

- ○ Forward Euler has a higher order than Backward Euler.
- ○ Runge-Kutta schemes are explicit if the $a_{ij}$ coefficients of the Butcher array are zero for all entries along and above the diagonal.
- ○ Backward Euler is implicit and second order accurate.
- ☒ Heun's method is second order accurate and explicit.
- ☒ The Crank-Nicolson method is implicit.
- ○ The Crank-Nicholson and Backward Euler methods are both second order accurate.

The Forward and backward Euler have the same order.
If the maxtrix $A$ is lower triangular it is explicit, if non-zero entries are above the diagonal it is implicit.
The backward Euler is implicit, but is first order accurate.

**Question 12**:
What are the values of approximation $u_1$ and $u_2$ using two iterations of the Backward Euler method for the ordinary differential equation $y' = 2y - 2$ with initial condition $y(0) = 1$ and step size $h = 0.1$.

- ○ $u_2 = 2$
- ○ $u_1 = -1$
- ☒ $u_1 = 1$
- ○ $u_1 = 1.5$
- ☒ $u_2 = 1$
- ○ $u_2 = 0.5$

From the Backward Euler formula:

$$u_{n+1} = u_n + h \cdot f(u_{n+1}, t_{n+1})$$

where $f(y, t) = 2y - 2$ with the initial condition

$$u_0 = 1$$

The first time step yields $u_1$ as

$$\begin{aligned} u_1 &= u_0 + h \cdot (2u_1 - 2), \\ u_1 &= 1 + 0.1 \cdot (2u_1 - 2), \\ u_1 &= 1 + 0.2u_1 - 0.2, \\ u_1 - 0.2u_1 &= 0.8, \\ 0.8u_1 &= 0.8, \\ u_1 &= 1 \end{aligned}$$

Thus, the values of $u_1$ and $u_2$ after two iterations of the Backward Euler method with a step size $h = 0.1$ are both 1.

**Question 13**:
Given the following data:

| $i$ | 0 | 1 | 2 |
|---|---|---|---|
| $x_i$ | 0 | 2 | 4 |
| $p_i$ | 2 | 1 | 2 |

Using Newton interpolation, which is the right collocation matrix?

- ○ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}$
- ✗ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 16 \end{pmatrix}$

Newton's interpolating polynomial can be written as:

$$p(x) = \alpha_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1)$$

where $\alpha_0$, $\alpha_1$ and $\alpha_2$ are the coefficients determined from the divided differences of the data points.

Given the data points the first step is to compute divided differences. For the first term:

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} = \frac{1 - 2}{2 - 0} = -\frac{1}{2}$$

and

$$f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{2 - 1}{4 - 2} = \frac{1}{2}.$$

Second divided difference:

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{0.5 - (-0.5)}{4 - 0} = \frac{1}{4}.$$

Formulate Newton's interpolating polynomial, from the divided differences:

$$\alpha_0 = f(x_0) = 2,$$
$$\alpha_1 = f[x_0, x_1] = -\frac{1}{2},$$
$$\alpha_2 = f[x_0, x_1, x_2] = \frac{1}{4}$$

From the data points and the polynomial, it is evident that the collocation matrix satisfies:

$$\begin{matrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{matrix} \quad \begin{matrix} 2 \\ -1/2 \\ 1/4 \end{matrix} = \begin{matrix} 2 \\ 1 \\ 2 \end{matrix}.$$