Dr. D. Sinden

CA-MATH-804: Numerical Analysis

Exam: Wednesday 25 May 2022

Question 1 [20 Points]: On the scaled unit square $\Omega = h[0,1]^2$, $h \in (0,1)$, consider the partial differential equation

$$\begin{aligned} -\left(\partial_{xx} u\left(x,y\right) + 2\partial_{yy} u\left(x,y\right)\right) &= f\left(x,y\right) && \text{in} \quad \Omega, \\ u\left(x,y\right) &= 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

Show that the weak form takes the form $a\left(u,v\right) = \left(f,v\right)$, where

$$a\left(u,v\right) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,\mathrm{d}\Omega$$

and

$$\left(f,v\right) = \int_\Omega f v\,\mathrm{d}\Omega.$$

What conditions are imposed on the test function $v\left(x,y\right)$?

Question 2 [20 Points]: For given v, $w \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ symmetric positive definite consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ where $\varphi\left(\alpha\right) = \left\|v + \alpha w\right\|_A^2$ and $\left\|\cdot\right\|_A$ is the energy norm given by $\left\|x\right\|_A = \sqrt{x \cdot Ax}$. Find α such that φ becomes minimal.
If A is a diagonal matrix D, show that $\left\|x\right\|_{D^2}^2 = \left\|Dx\right\|_2^2$.

Question 3 [20 Points]: Using

$$D^+ f = \frac{f\left(x+h\right) - f\left(x\right)}{h},$$

and

$$D^- f = \frac{f\left(x\right) - f\left(x-h\right)}{h}$$

derive the discrete approximation to the second-order derivative as a matrix $A \in \mathbb{R}^{n\times n}$, when $u_0 = 0$ and $u_{n+1} = 0$.
The matrix has eigenvalues

$$\lambda_j = \frac{2}{h^2}\left(1 + \cos\left(\frac{\pi j}{n+1}\right)\right) \quad \text{for} \quad j = 1, \ldots, n.$$

Using the half-angle formula

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2},$$

find the ratio of the maximum and minimum eigenvalues.

Question 4 [20 Points]: Show that a solution which minimizes

$$\Phi(y) = \frac{1}{2} y \cdot Ay - y \cdot b$$

also solves the linear system $Ax = b$. Define the residue $r^{(k)}$ for an iterative scheme which solves the linear system. Show that $\nabla\Phi\left(x^{(k)}\right) = -r^{(k)}$. Consider the iterative scheme

$$x^{(k+1)} = x^{(k)} + \alpha^{(k)} d^{(k)}$$

and set $d^{(k)} = r^{(k)}$. By considering the minimum of $\Phi\left(x^{(k+1)}\right)$, i.e. $\Phi\left(x^{(k)} + \alpha^{(k)} d^{(k)}\right)$, with respect to α, show that

$$\alpha^{(k)} = \frac{r^{(k)} \cdot r^{(k)}}{r^{(k)} \cdot Ar^{(k)}}.$$

Question 5 [20 Points]: Show, by deriving the weights of the quadrature scheme using the Lagrange interpolating polynomials defined via

$$l_i(x) = \prod_{\substack{j=1, \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

that the scheme

$$I(f) = \sum_{i=1}^{3} \alpha_i f(x_i) = \frac{2}{3}\left[2f\left(-\frac{1}{2}\right) - f(0) + 2f\left(\frac{1}{2}\right)\right]$$

where $\alpha_i = \int_{-1}^{1} l_i(x)\, \mathrm{d}x$, is a Lagrange quadrature formula for 3 nodes $x_1 = -\frac{1}{2}$, $x_2 = 0$ and $x_3 = \frac{1}{2}$ on the interval $[-1, 1]$.
Determine the quadrature error of $I(f)$.

Question 6 [20 Points]: Consider the function $H : \mathbb{R} \to \mathbb{R}$ with

$$H(x) = \begin{cases} 0 & \text{for} \quad x \leq 0, \\ 1 & \text{else.} \end{cases}$$

Show that this function has a distributional derivative.