Jacobs University

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CA-MATH-804: Numerical Analysis

Exam & Solutions

All trigonometric values are in radians.

Question 1 [20 Points]: On the scaled unit square $\Omega = h[0,1]^2$, $h \in (0,1)$, consider the partial differential equation

$$\begin{aligned} -\left(\partial_{xx} u\left(x,y\right) + 2\partial_{yy} u\left(x,y\right)\right) &= f\left(x,y\right) && \text{in} \quad \Omega, \\ u\left(x,y\right) &= 0 && \text{on} \quad \partial\Omega. \end{aligned}$$

Show that the weak form takes the form $a\left(u,v\right) = \left(f,v\right)$, where

$$a\left(u,v\right) = \int_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}\,\mathrm{d}\Omega$$

and

$$\left(f,v\right) = \int_\Omega f v \,\mathrm{d}\Omega.$$

What conditions are imposed on the test function $v\left(x,y\right)$?

Let $\Omega = h[0,1]^2 \subset \mathbb{R}^2$ and consider

$$-\big(u_{xx}(x,y) + 2\,u_{yy}(x,y)\big) = f(x,y) \quad \text{in } \Omega, \quad \text{with} \quad u = 0 \quad \text{on } \partial\Omega.$$

Multiply by a test function v and integrate over Ω:

$$\int_\Omega \big(-u_{xx} - 2\,u_{yy}\big)\,v\,\mathrm{d}\Omega = \int_\Omega f\,v\,\mathrm{d}\Omega.$$

Integrate by parts and use $v = 0$ on $\partial\Omega$ to kill the boundary terms:

$$\int_\Omega u_{xx}\,v\,\mathrm{d}\Omega = -\int_\Omega u_x\,v_x\,\mathrm{d}\Omega, \qquad \int_\Omega u_{yy}\,v\,\mathrm{d}\Omega = -\int_\Omega u_y\,v_y\,\mathrm{d}\Omega.$$

Hence the weak form is

$$a(u,v) = (f,v), \quad \text{where} \quad a(u,v) = \int_\Omega \big(u_x\,v_x + 2\,u_y\,v_y\big)\,\mathrm{d}\Omega, \quad (f,v) = \int_\Omega f\,v\,\mathrm{d}\Omega.$$

The test functions v are taken from

$$H_0^1(\Omega) = \big\{v \in H^1(\Omega) \;:\; v = 0 \text{ on } \partial\Omega\big\}.$$

Question 2 [20 Points]: For given v, $w \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ symmetric positive definite consider the function $\varphi : \mathbb{R} \to \mathbb{R}$ where $\varphi(\alpha) = \|v + \alpha w\|_A^2$ and $\|\cdot\|_A$ is the energy norm given by $\|x\|_A = \sqrt{x \cdot Ax}$. Find α such that φ becomes minimal.
If A is a diagonal matrix D, show that $\|x\|_{D^2}^2 = \|Dx\|_2^2$.

The energy-norm expands as

$$\begin{aligned}\phi(\alpha) &= \|v + \alpha w\|_A^2 \\ &= (v + \alpha w)^T A\,(v + \alpha w) \\ &= v^T Av + 2\alpha\, v^T Aw + \alpha^2\, w^T Aw.\end{aligned}$$

Differentiating w.r.t. α and setting to zero gives

$$\phi'(\alpha) = 2\, v^T Aw + 2\alpha\, w^T Aw \implies \alpha = -\frac{v^T Aw}{w^T Aw}.$$

Hence the minimizer is

$$\alpha^* = -\frac{v^T A\, w}{w^T A\, w}.$$

If A is a diagonal matrix D,

$$\begin{aligned}\|x\|_{D^2}^2 &= x^T D^2 x \\ &= x^T D^T D\, x \\ &= (D\, x)^T (D\, x) \\ &= \|D\, x\|_2^2.\end{aligned}$$

Using the half-angle identity

$$1 + \cos\theta = 2\cos^2\frac{\theta}{2},$$

then

$$\lambda_j = \frac{2}{h^2}\Big(1 + \cos\frac{\pi j}{n+1}\Big) = \frac{4}{h^2}\cos^2\Big(\tfrac{\pi j}{2(n+1)}\Big).$$

Thus

$$\lambda_{\max} = \lambda_1 = \frac{4}{h^2}\cos^2\Big(\tfrac{\pi}{2(n+1)}\Big), \qquad \lambda_{\min} = \lambda_n = \frac{4}{h^2}\cos^2\Big(\tfrac{n\pi}{2(n+1)}\Big) = \frac{4}{h^2}\sin^2\Big(\tfrac{\pi}{2(n+1)}\Big).$$

Hence the spectral ratio is

$$\frac{\lambda_{\max}}{\lambda_{\min}} = \frac{\cos^2\big(\frac{\pi}{2(n+1)}\big)}{\sin^2\big(\frac{\pi}{2(n+1)}\big)} = \cot^2\Big(\tfrac{\pi}{2(n+1)}\Big).$$

Question 3 [20 Points]: Using

$$D^{+}f = \frac{f(x+h) - f(x)}{h},$$

and

$$D^{-}f = \frac{f(x) - f(x-h)}{h}$$

derive the discrete approximation to the second-order derivative as a matrix $A \in \mathbb{R}^{n\times n}$, when $u_0 = 0$ and $u_{n+1} = 0$.
The matrix has eigenvalues

$$\lambda_j = \frac{2}{h^2}\left(1 + \cos\left(\frac{\pi j}{n+1}\right)\right) \quad \text{for} \quad j = 1, \ldots, n.$$

Using the half-angle formula

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2},$$

find the ratio of the maximum and minimum eigenvalues.

Let $h = \frac{1}{n+1}$, and $x_i = i\,h,\ i = 0, 1, \ldots, n+1$, with $u_0 = u_{n+1} = 0$.

$$D_{+}u_i = \frac{u_{i+1} - u_i}{h}, \quad D_{-}u_i = \frac{u_i - u_{i-1}}{h},$$

So that Discrete second derivative by backward of forwards (or vice versa):

$$D_{-}D_{+}u_i = \frac{1}{h}\Big(D_{+}u_i - D_{+}u_{i-1}\Big) = \frac{u_{i-1} - 2u_i + u_{i+1}}{h^2}.$$

Hence the discrete approximation to $u'' = f$ leads to

$$\frac{u_{i-1} - 2u_i + u_{i+1}}{h^2} = f_i, \quad i = 1, \ldots, n.$$

In matrix form $A\,u = f$, where

$$A = \frac{1}{h^2}\begin{pmatrix} -2 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ 0 & & & 1 & -2 \end{pmatrix} \in \mathbb{R}^{n\times n}.$$

With this A and the Dirichlet boundary conditions $u_0 = u_{n+1} = 0$, the finite-difference approximation to $u = f$ is simply $Au = f$.

Question 4 [20 Points]: Show that a solution which minimizes

$$\Phi(y) = \frac{1}{2} y \cdot Ay - y \cdot b$$

also solves the linear system $Ax = b$. Define the residue $r^{(k)}$ for an iterative scheme which solves the linear system. Show that $\nabla \Phi\left(x^{(k)}\right) = -r^{(k)}$. Consider the iterative scheme

$$x^{(k+1)} = x^{(k)} + \alpha^{(k)} d^{(k)}$$

and set $d^{(k)} = r^{(k)}$. By considering the minimum of $\Phi\left(x^{(k+1)}\right)$, i.e. $\Phi\left(x^{(k)} + \alpha^{(k)} d^{(k)}\right)$, with respect to α, show that

$$\alpha^{(k)} = \frac{r^{(k)} \cdot r^{(k)}}{r^{(k)} \cdot Ar^{(k)}}.$$

Note assume A is symmetric (so that $\nabla(\frac{1}{2} y^T A y) = A y$). Then, to show that minimiser of

$$\Phi(y) = \tfrac{1}{2} y^T A y - y^T b$$

satisfies $A y = b$. first compute the gradient $\nabla\Phi(y) = A y - b$. and show that stationarity $\nabla\Phi(y) = 0$ gives $A y = b$.

Define the residual at step k by

$$r^{(k)} = b - A x^{(k)}.$$

Then

$$\nabla\Phi\big(x^{(k)}\big) = A x^{(k)} - b = -r^{(k)}.$$

For the scheme

$$x^{(k+1)} = x^{(k)} + \alpha^{(k)} d^{(k)}, \quad d^{(k)} = r^{(k)},$$

consider the one-variable function

$$\varphi(\alpha) = \Phi\big(x^{(k)} + \alpha r^{(k)}\big).$$

Expand and differentiate:

$$\varphi(\alpha) = \tfrac{1}{2} (x^{(k)} + \alpha r^{(k)})^T A (x^{(k)} + \alpha r^{(k)}) - (x^{(k)} + \alpha r^{(k)})^T b,$$

and

$$\begin{aligned}\varphi'(\alpha) &= (x^{(k)})^T A r^{(k)} + \alpha (r^{(k)})^T A r^{(k)} - (r^{(k)})^T b \\ &= -\left(r^{(k)}\right)^T r^{(k)} + \alpha \left(r^{(k)}\right)^T A r^{(k)}.\end{aligned}$$

Setting $\varphi'(\alpha) = 0$ yields

$$\alpha^{(k)} = \frac{\left(r^{(k)}\right)^T r^{(k)}}{\left(r^{(k)}\right)^T A r^{(k)}}.$$

Hence the optimal step-size is

$$\alpha^{(k)} = \frac{r^{(k)} \cdot r^{(k)}}{r^{(k)} \cdot A r^{(k)}}.$$

Question 5 [20 Points]: Show, by deriving the weights of the quadrature scheme using the Lagrange interpolating polynomials defined via

$$l_i(x) = \prod_{\substack{j=1,\\ j\neq i}}^{n} \frac{x - x_j}{x_i - x_j}$$

that the scheme

$$I(f) = \sum_{i=1}^{3} \alpha_i f(x_i) = \frac{2}{3}\left[2f\left(-\frac{1}{2}\right) - f(0) + 2f\left(\frac{1}{2}\right)\right]$$

where $\alpha_i = \int_{-1}^{1} l_i(x)\,\mathrm{d}x$, is a Lagrange quadrature formula for 3 nodes $x_1 = -\frac{1}{2}$, $x_2 = 0$ and $x_3 = \frac{1}{2}$ on the interval $[-1, 1]$.
Determine the quadrature error of $I(f)$.

To approximate

$$I(f) = \int_{-1}^{1} f(x)\,\mathrm{d}x$$

by a three-point Lagrange-quadrature formula with nodes

$$x_1 = -\tfrac{1}{2}, \quad x_2 = 0, \quad x_3 = +\tfrac{1}{2},$$

let

$$\ell_i(x) = \prod_{\substack{j=1\\ j\neq i}}^{3} \frac{x - x_j}{x_i - x_j}, \qquad i = 1, 2, 3,$$

be the Lagrange basis polynomials. Then the weights are

$$\alpha_i = \int_{-1}^{1} \ell_i(x)\,\mathrm{d}x, \qquad i = 1, 2, 3,$$

and the quadrature reads

$$I(f) \approx \sum_{i=1}^{3} \alpha_i f(x_i).$$

For the computation of the weights,

$$\begin{aligned}
\ell_1(x) &= \frac{(x-0)(x-\frac{1}{2})}{(-\frac{1}{2}-0)(-\frac{1}{2}-\frac{1}{2})} \\
&= \frac{(x)(x-\frac{1}{2})}{\frac{1}{4}} \\
&= 4x\,(x-\tfrac{1}{2}), \\
\ell_2(x) &= \frac{(x+\frac{1}{2})(x-\frac{1}{2})}{(0+\frac{1}{2})(0-\frac{1}{2})} \\
&= -4\,(x^2-\tfrac{1}{4}), \\
\ell_3(x) &= \frac{(x+\frac{1}{2})(x-0)}{(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-0)} \\
&= \frac{(x+\frac{1}{2})\,x}{\frac{1}{4}} \\
&= 4x\,(x+\tfrac{1}{2}).
\end{aligned}$$

Hence

$$\alpha_1 = \int_{-1}^{1} 4x\,(x-\tfrac{1}{2})\,\mathrm{d}x = \frac{4}{3}, \quad \alpha_2 = \int_{-1}^{1} -4\big(x^2-\tfrac{1}{4}\big)\,\mathrm{d}x = -\frac{2}{3}, \quad \alpha_3 = \int_{-1}^{1} 4x\,(x+\tfrac{1}{2})\,\mathrm{d}x = \frac{4}{3}.$$

One then checks easily that

$$\sum_{i=1}^{3} \alpha_i = 2, \quad \sum_{i=1}^{3} \alpha_i \, x_i = 0, \quad \sum_{i=1}^{3} \alpha_i \, x_i^2 = \tfrac{2}{3}, \quad \sum_{i=1}^{3} \alpha_i \, x_i^3 = 0,$$

so the rule is exact for all polynomials of degree up to 3. In the compact form

$$I(f) \approx \frac{2}{3}\Big[\, 2f(-\tfrac{1}{2}) \; - \; f(0) \; + \; 2f(\tfrac{1}{2})\Big].$$

To compute the error term, note that since the rule integrates exactly all polynomials of degree ≤ 3, the first non-zero error appears at degree 4. A standard remainder-of-Lagrange-interpolation argument gives: for some $\xi \in (-1, 1)$,

$$E(f) = \int_{-1}^{1} f(x)\, \mathrm{d}x \; - \; \sum_{i=1}^{3} \alpha_i \, f(x_i) = \frac{f^{(4)}(\xi)}{4!} \int_{-1}^{1} (x - x_1)(x - x_2)(x - x_3)\, \mathrm{d}x.$$

A direct evaluation of the integral shows

$$\int_{-1}^{1} (x + \tfrac{1}{2})\, x \,(x - \tfrac{1}{2})\, \mathrm{d}x = \frac{7}{30},$$

and hence

$$E(f) = \frac{f^{(4)}(\xi)}{24} \, \frac{7}{30} = \frac{7}{720}\, f^{(4)}(\xi).$$

Thus the three-point Lagrange quadrature on $[-1, 1]$ with nodes $\{-\tfrac{1}{2}, 0, \tfrac{1}{2}\}$ has error

$$\int_{-1}^{1} f(x)\, \mathrm{d}x - \frac{2}{3}\Big[2f(-\tfrac{1}{2}) - f(0) + 2f(\tfrac{1}{2})\Big] = \frac{7}{720}\, f^{(4)}(\xi).$$

Question 6 [20 Points]: Consider the function $H : \mathbb{R} \to \mathbb{R}$ with

$$H(x) = \begin{cases} 0 & \text{for} \quad x \le 0, \\ 1 & \text{else.} \end{cases}$$

Show that this function has a distributional derivative.

Define the Heaviside function $H : \mathbb{R} \mapsto \{0, 1\}$ by

$$H(x) = \begin{cases} 0, & x < 0, \\ 1, & x \ge 0. \end{cases}$$

As a distribution, for any test function $\varphi(x) \in C_c^\infty(\mathbb{R})$,

$$\begin{aligned} \langle H, \varphi \rangle &= \int_{\mathbb{R}} H(x)\,\varphi(x)\,\mathrm{d}x \\ &= \int_0^\infty \varphi(x)\,\mathrm{d}x. \end{aligned}$$

The distributional derivative $H'(x)$ is defined by

$$\begin{aligned} \langle H', \psi \rangle &= -\langle H, \psi' \rangle \\ &= -\int_0^\infty \psi'(x)\,\mathrm{d}x \\ &= -\big[\psi(x)\big]_0^\infty \\ &= -\big(0 - \psi(0)\big) \\ &= \psi(0). \end{aligned}$$

Since $\langle \delta, \psi \rangle = \psi(0)$, it follows that

$$H'(x) = \delta(x)$$

in the sense of distributions.