Dr. D. Sinden

CTMS-MAT-13: Numerical Methods

Exam: Friday 22 August 2025

All questions carry equal marks. Only five questions will be marked. Please use the booklet provided, clearly indicating in the inside cover which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

(a) What three operations form the elementary row operations used in Gaussian elimination? [3]

(b) Given $a = 2$, find the solution to the linear system $A\vec{x} = \vec{b}$ with

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 3 & -1 & 3 \\ 0 & a & 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

[7]

(c) For the matrix A, when $a = 2$, what rows should be swapped in order to perform partial pivoting on the first column? [3]

(d) Show that when $a = \dfrac{13}{12}$ the linear system has no unique solution. [7]

Question 2:

Consider the integral

$$I = \int_0^1 e^{2x+1}\,\mathrm{d}x.$$

Let the integral be discretized into subintervals of width $h = 1/2^k$, where $k = 1, 2$ and 3 so that n, the number of intervals, is $n = 2, 4$ and 8. The number of points for each discretization is given by $m = n + 1$.
Noting that the Trapezium rule is given by

$$I_n = \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{m-1} f(x_i) + f(x_m)\right)$$

and Simpson's rule is given by

$$I_n = \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{m/2} f(x_{2i-1}) + 2\sum_{i=1}^{m/2-1} f(x_{2i}) + f(x_m)\right).$$

(a) Compute the first three approximations using Trapezium rule. [8]

(b) Compute the first three approximations using Simpson's rule. [8]

(c) For the Trapezium rule, the error is proportional to which power of h? [2]

(d) By which process can Simpson's rule be derived from the Trapezium rule? [2]

Question 3:

(a) What is an upper triangular matrix? [2]

(b) For LU decomposition of a matrix A, the matrix A must be invertible. What condition on the determinant is necessary for a matrix to be invertible? [2]

(c) What condition on the eigenvalues of A is necessary for Cholseky decomposition to be performed? [2]

(d) If the entries of the matrix are real numbers, what addition condition on the matrix A is necessary for Cholseky decomposition to be performed? [2]

(e) Find the lower triangular Cholesky matrix $\tilde{L}$, such that $A = \tilde{L}\tilde{L}^T$, for

$$A = \begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix} \quad \text{with} \quad \tilde{L} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}.$$

[12]

Question 4:

Consider the measurement values $p_0 = 5, p_1 = 4$, and $p_2 = 6$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$.

Let the function $p(u) = a\cos(u) + bu = \sum_{k=0}^{1} \beta_k \varphi_k(u)$ approximate the data in the least squares sense, where $\beta_0 = a$ and $\beta_1 = b$, and $\varphi_0 = \cos(u)$ and $\varphi_1 = u$.

(a) By considering

$$\frac{\partial E}{\partial \beta_j} = \sum_{i=0}^{n} \left(p_i - \sum_{k=0}^{1} \beta_k \varphi_k(u_i) \right) \varphi_j(u_i) = 0,$$

show that the normal equations are given by

$$\begin{aligned} a\sum_{i=0}^{2} \cos^2(u_i) + b\sum_{i=0}^{2} u_i \cos(u_i) &= \sum_{i=0}^{2} p_i \cos(u_i), \\ a\sum_{i=0}^{2} u_i \cos(u_i) + b\sum_{i=0}^{2} u_i^2 &= \sum_{i=0}^{2} p_i u_i. \end{aligned}$$

[6]

(b) Write this as a linear equation and solve for a and b, showing

$$a = \left(25 + 2\sqrt{2}\right)/7 \quad \text{and} \quad b = \left(88 - 10\sqrt{2}\right)/(7\pi).$$

[7]

(c) Compute numerically the error which is minimized, and show that it can be expressed as

$$E = \frac{2}{7}\left(27 - 10\sqrt{2}\right).$$

[7]

Question 5:

Newton's method is given by in a general form as

$$\vec{x}_{n+1} = \vec{x}_n - J^{-1}(\vec{x}_n)\,\vec{f}(\vec{x}_n)$$

where J is the Jacobian matrix of $\vec{f}$. What is the second iterate, $\vec{x}_2$, for the function

$$\vec{f}(\vec{x}) = \begin{pmatrix} 2u - v + \frac{1}{9}e^{-u} - 1 \\ -u + 2v + \frac{1}{9}e^{-v} \end{pmatrix}$$

where $\vec{x} = (u, v)^T$ with initial guess $\vec{x}_0 = (1, 1)^T$?

- ○ (-0.5972, -0.2556)
- ○ (0.2556, 0.5972)
- ○ (0.5972, -0.2556)
- ○ (-0.5972, 0.2556)
- ○ (0.5972, 0.2556)
- ○ (0.2556, -0.5972)

Question 6:

Given the following data which fits $y = \sin(x)$,

i	0	1	2
x_i	0	$\frac{\pi}{2}$	π
y_i	0	1	0

Using any polynomial interpolation, such as Lagrange polynomials, which are given by

$$p(x) = \sum_{i=0}^{N} y_i\, l_i(x), \quad \text{where} \quad l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{x - x_j}{x_i - x_j},$$

derive a quadratic approximation. What is the co-efficient of the quadratic term?

- ○ $4/\pi$
- ○ $-4/\pi^2$
- ○ $4/\pi^2$
- ○ $3/\pi^2$
- ○ $2/\pi^2$
- ○ $-1/\pi^2$

Question 7:

The ordinary differential equation

$$y'(t) = \left(1 - \frac{y}{K}\right) y \quad \text{with} \quad y(0) = 1$$

has the solution, $y(t) = \dfrac{y_0 e^t}{1 + y_0 e^t / K}$. Let $K = 2$, then using the forward Euler method, i.e.

$$u_{n+1} = u_n + h f\left(t_n, u_n\right)$$

with step size $h = 0.1$, show the global truncation error $|y(2h) - u_2|$ after two steps is

- ○ -0.935662
- ○ 0.000288
- ○ 0.0071290
- ○ 0.341875
- ○ 0.05656
- ○ 0.189544

Question 8:

The Jacobi scheme solves the linear equation $A\vec{x} = \vec{b}$ iteratively as

$$\vec{x}_{n+1} = \left(I - D^{-1}A\right)\vec{x}_n + D^{-1}\vec{b}$$

where D is the diagonal matrix of A. Find the residual, $|\vec{x}^* - \vec{x}_2|$ between the exact solution, $\vec{x}^*$, and the Jacobi solution after two iterates with initial guess $\vec{x}_0 = (1, 1)^T$, where

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- ○ 1/4
- ○ 1/5
- ○ 2/5
- ○ 1/2
- ○ 3/4
- ○ 0