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CTMS-MAT-13: Numerical Methods

Exam: Saturday 29 August 2026

All questions carry equal marks. Only five questions will be marked. Please use the booklet provided, clearly indicating in the inside cover which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

(a) Give one criterion for a square matrix to be *singular*. [2]

(b) If a matrix is singular, what does this imply about the number of solutions of $A\vec{x} = \vec{b}$? [2]

(c) Given the matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ -3 & -1 & 2 \\ -2 & 1 & 2 \end{pmatrix},$$

use Gaussian elimination to reduce A to row echelon (upper triangular) form U. [7]

(d) By applying Gaussian elimination, or any other method, show that the solution to $A\vec{x} = \vec{b}$ with

$$\vec{b} = \begin{pmatrix} 8 \\ -11 \\ -3 \end{pmatrix} \quad \text{is} \quad \vec{x} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}.$$

[7]

(e) For an $n \times n$ invertible matrix, state the order (in n) of the number of arithmetic operations required to reduce the matrix to upper triangular form by Gaussian elimination. [2]

Question 2:

(a) Find the Jacobian matrix $J = J(x, y)$ for the vector-valued function

$$\vec{f}(x, y) = \begin{pmatrix} x^2 + y^2 - 4 \\ xy - 1 \end{pmatrix}.$$

[4]

(b) Show that the inverse of the Jacobian is

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} x & -2y \\ -y & 2x \end{pmatrix}, \qquad \text{where} \quad |J| = 2x^2 - 2y^2.$$

[6]

(c) Let $\vec{u}_n = (x_n, y_n)^T$. Using Newton's method, $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n)\,\vec{f}(\vec{u}_n)$, with initial guess $\vec{u}_0 = (2, 1)^T$, show that the first iterate is $\vec{u}_1 = (2, \frac{1}{2})^T$ and compute the second iterate $\vec{u}_2$. [10]

Question 3:

Consider the integral

$$I = \int_0^1 e^x \,\mathrm{d}x = e - 1 = 1.718281828459045.$$

(a) Given the Trapezium rule

$$I_n = \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-1} f(x_i)\right), \qquad h = \frac{b-a}{n},$$

show that the approximations for $n = 2^k$, $k = 0, 1, 2$, are

k	n	I_n
0	1	1.85914091
1	2	1.75393109
2	4	1.72722190

[8]

(b) Noting the Trapezium rule has error behaviour $I = I_n + a_1 h^2 + a_2 h^4 + \cdots$, and considering the difference of the errors for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3}\left(4R_k^0 - R_{k-1}^0\right), \qquad R_0^0 = I_1,\ R_1^0 = I_2,\ \ldots$$

[6]

(c) Using the values above as $R_k^0 = I_{2^k}$, compute R_1^1, R_2^1 and hence, using $R_2^2 = \frac{1}{15}\left(16R_2^1 - R_1^1\right)$, show the second Romberg extrapolation is given by $R_2^2 = 1.718283$. [6]

Question 4:

(a) For the numerical solution of ordinary differential equations $y' = f(y,t)$, explain the difference between an *explicit* and an *implicit* method. [3]

(b) For the second-order linear ordinary differential equation

$$y''(t) = -3y'(t) - 2y(t) \qquad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0,$$

write the equation as a first-order system in $\vec{u} = (y, y')^T$, i.e. $\vec{u}' = A\vec{u}$, and give the matrix A. [4]

(c) Show that the forward Euler scheme can be written $\vec{u}_{n+1} = (I + hA)\vec{u}_n$, and hence compute the first two steps with $h = 0.1$, i.e. $\vec{u}_1$ and $\vec{u}_2$. [7]

(d) Show that the backward Euler scheme yields $\vec{u}_{n+1} = (I - hA)^{-1}\vec{u}_n$, and compute the first step $\vec{u}_1$ with $h = 0.1$. [6]

Question 5:

Given the matrix

$$A = \begin{pmatrix} 16 & 4 & 8 \\ 4 & 5 & -4 \\ 8 & -4 & 22 \end{pmatrix},$$

what is the Cholesky matrix L (with $A = LL^T$) for the matrix A?

○ $\begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -3 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 4 & 1 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -3 & 4 \end{pmatrix}$

○ $\begin{pmatrix} 4 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & 3 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 8 & 0 & 0 \\ 1 & 2 & 0 \\ 2 & -3 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 4 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & -3 & 3 \end{pmatrix}$

Question 6:

Given the following data:

i	0	1	2
x_i	0	1	3
y_i	1	2	1

Using polynomial interpolation, such as Lagrange or Newton interpolation, what is the value of $y(2)$?

- ○ 3/2
- ○ 7/3
- ○ 11/6
- ○ 2
- ○ 5/2
- ○ 1

Question 7:

Using the Jacobi scheme $\vec{x}_{n+1} = (I - D^{-1}A)\vec{x}_n + D^{-1}\vec{b}$, where D is the diagonal of A, find the residual $\|\vec{x}^* - \vec{x}_2\|_2$ (the Euclidean, or L^2, norm) between the exact solution $\vec{x}^*$ and the second Jacobi iterate $\vec{x}_2$, starting from $\vec{x}_0 = (0,0)^T$, where

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \qquad \text{and} \qquad \vec{b} = \begin{pmatrix} 5 \\ 5 \end{pmatrix}.$$

- ○ $\frac{5\sqrt{2}}{36}$
- ○ $\frac{5\sqrt{2}}{18}$
- ○ $\frac{25\sqrt{2}}{36}$
- ○ $\frac{5}{36}$
- ○ $\frac{5}{18}$
- ○ $\frac{5}{9}$

Question 8:

The differential equation

$$y'(t) = y - t^2 + 1 \qquad \text{with} \quad y(0) = \tfrac{1}{2}$$

has the exact solution $y(t) = (t+1)^2 - \frac{1}{2}e^t$. From the classical explicit Runge–Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array}$$

where

$$u_{k+1} = u_k + h\sum_{i=1}^{4} b_i f\left(u_k + h\sum_{j=1}^{4} a_{i,j}k_j, t_k + c_i h\right) = u_k + h\sum_{i=1}^{4} b_i k_i$$

and using step size $h = 0.2$, what is $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 1.1×10^{-5}
- ○ 1.4×10^{-2}
- ○ 5.8×10^{-4}
- ○ 2.3×10^{-3}
- ○ 3.7×10^{-1}
- ○ 0