Dr. D. Sinden

JTMS-MAT-13: Numerical Methods

Exam: Thursday 23 May 2024

All questions carry equal marks. Answer 5 questions only. Please use the booklet provided.

All trigonometric values are in radians.

Question 1:

(a) For the function $f(x) = (x + 1/2)\cos(x)$, show that Newton's method is

$$x_{k+1} = x_k - \frac{(x_k + 1/2)\cos(x_k)}{\cos(x_k) - (x_k + 1/2)\sin(x_k)}.$$

(b) From the definition of Newton's method, show that the secant method is given by

$$x_{k+1} = x_k - f(x_k)\frac{x_{k-1} - x_k}{f(x_{k-1}) - f(x_k)}.$$

(c) Compute two iterates of the Newton method with initial guess $x_0 = -1/3$.

(d) Compute two iterates of the secant method with initial guess $x_0 = -1/3$ and $x_1 = -0.4$.

Question 2:

Consider the linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

(a) By converting this 2^{nd} order ordinary differential equation into a system of two coupled first order ODEs, one in $y(t)$ and one in $y'(t)$, show the system can be written as a vector-valued ordinary differential equation in $\vec{v}(t) = (y(t), y'(t))^T$, in the form $f(\vec{v}) = A\vec{v}$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}.$$

(b) Show that the forward Euler method can be written as

$$\vec{u}_{n+1} = (I + hA)\vec{u}_n$$

for some approximation $\vec{u} \in \mathbb{R}^2$, and provide the full system for $\vec{u}_{n+1}$ for the ODE presented above.

(c) Show that the backward Euler method yields

$$\vec{u}_{n+1} = \frac{1}{1 + 4h - h^2}\begin{pmatrix} 1 + 4h & -h \\ -h & 1 \end{pmatrix}\vec{u}_n.$$

(d) Calculate approximations $y(0.3)$ and $y'(0.3)$ using forward Euler method with $h = 0.15$.

(e) Calculate $y(0.3)$ and $y'(0.3)$ using backward Euler method with $h = 0.15$.

Question 3:

(a) What is meant if a matrix is said to be diagonally dominant?

(b) Give a definition for a positive definite matrix.

(c) What is the range of ω for which the method of successive over relaxation converges for a semi-positive definite matrix?

(d) Show that the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

has eigenvalues $\lambda = 4$, $4 + \sqrt{10}$ and $4 - \sqrt{10}$, and thus is can be inverted using the method of successive over relaxation.

Question 4:

(a) Show, by constructing an cubic polynomial, $q(x)$, which is orthogonal to 1, x and x^2, i.e.

$$\int_{-1}^{1} x^i q(x)\,\mathrm{d}x = 0 \quad \text{for} \quad i = 0, 1, 2$$

that the Gauss nodes for Gaussian quadrature are $x_i^* = -\sqrt{\frac{3}{5}}$, 0, $\sqrt{\frac{3}{5}}$.

(b) Show that the Lagrange polynomials for the function

$$f(x) = 3x\cos(x)$$

which generates the data

i	0	1	2
x_i	$-\pi/4$	0	$\pi/4$
y_i	$-\dfrac{3\pi}{4\sqrt{2}}$	0	$\dfrac{3\pi}{4\sqrt{2}}$

are

$$l_0 = \frac{8}{\pi^2}x\left(x - \pi/4\right), \quad l_1 = -\frac{16}{\pi^2}\left(x^2 - \pi^2/16\right) \quad \text{and} \quad l_2 = \frac{8}{\pi^2}x\left(x + \pi/4\right).$$

(c) Thus, using $A_i = \int_{-1}^{1} l_i(x)\,\mathrm{d}x$, show that the approximation for the integral yields

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{2} A_i f\left(x_i^*\right) = 0.$$

Question 5: Given the following data:

i	0	1	2
x_i	0	1	3
y_i	1	3	2

Using polynomial interpolation, what is the value of $y(2)$?

- ○ $y(2) = 3/2$
- ○ $y(2) = 3/4$
- ○ $y(2) = 10/3$
- ○ $y(2) = 11/4$
- ○ $y(2) = 4/9$
- ○ $y(2) = 0$

Question 6: The differential equation

$$y'(t) = 1 - 4y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{4}\left(3e^{-4t} + 1\right)$. From the Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h \sum_{i=1}^{4} b_i f\left(u_k + h \sum_{j=1}^{4} a_{i,j} k_j, t_k + c_i h\right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 0.1
- ○ 0.449
- ○ 0
- ○ 0.5
- ○ 0.839
- ○ 0.164

Question 7: Given the integral

$$I = \int_{1/4}^{1/5} \frac{1}{2} + \sin(\pi x)\, \mathrm{d}x$$

what is the error of the approximate integral for the Trapezium rule when using five subintervals?

- ○ 2.30e-5
- ○ 1.44e-6
- ○ 2.11e-4
- ○ 1.67e-3
- ○ 2.67e-6
- ○ 1.44e-7

Question 8: Given the matrix

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix}$$

what is the Cholesky matrix L for the matrix A?

- ○ $\begin{pmatrix} 2 & 0 & 0 \\ 6 & 2 & 0 \\ -8 & 5 & 1 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & -4 \\ 3 & 8 & 1 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$
- ○ $\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 6 \end{pmatrix}$
- ○ $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ 7 & 9 & 1 \end{pmatrix}$