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CTMS-MAT-13: Numerical Methods

Exam: Wednesday 21 May 2025

All questions carry equal marks. Only five questions will be marked. Please use the booklet provided, clearly indicating in the inside cover which questions are to be marked.

All trigonometric values are in radians.

Question 1:

(a) Given Newton's method in one dimension is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

using a first-order approximation for the derivative, derive the secant method.

(b) Given a two-dimensional function, $\vec{f}(x,y) = (f_1(x,y), f_2(x,y))^T$, what is the Jacobian?

(c) For

$$\vec{f}(x,y) = \begin{pmatrix} x^2 + y^2 - xy \\ y^2 + e^{x+y} \end{pmatrix}$$

show the inverse of the Jacobian is given by

$$J^{-1}(x,y) = \frac{1}{\det(J)} \begin{pmatrix} 2y + e^{x+y} & x - 2y \\ -e^{x+y} & 2x - y \end{pmatrix}, \quad \text{where} \quad \det(J) = 3e^{x+y}(x-y) + 2y(2x-y).$$

(d) With initial condition $(x_0, y_0)^T = (1,3)^T$, find the first iterate of the Newton method.

Question 2:

(a) What is a singular matrix?

(b) Given the matrix

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 3 & 4 & 3 \\ 1 & 3 & 5 \end{pmatrix},$$

show the row echelon form of the matrix can be written as

$$U = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 7 & 9 \\ 0 & 0 & 52 \end{pmatrix}.$$

(c) By applying Gaussian elimination, or any other method, find the solution to the linear equation $A\vec{x} = \vec{b}$, where $\vec{b}$ is given by

$$\vec{b} = \begin{pmatrix} 61 \\ 63 \\ 43 \end{pmatrix}.$$

(d) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

(e) If an $n \times n$ matrix is in lower triangular form, what is the order of the upper limit for the number of arithmetic operations to perform back substitution?

Question 3:

(a) Show, by constructing an cubic polynomial, $q(x)$, which is orthogonal to 1, x and x^2, i.e.

$$\int_{-1}^{1} x^i \, q(x) \, \mathrm{d}x = 0 \quad \text{for} \quad i = 0, 1, 2$$

that the Gauss nodes for Gaussian quadrature are $x_i^* = -\sqrt{\frac{3}{5}}, 0, \sqrt{\frac{3}{5}}$.

(b) Lagrange polynomials are defined as

$$l_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

Derive the three Lagrange polynomials for the x_i^* given above.

(c) Show that the weights c_i such that

$$c_i \int_{-1}^{1} l_i(x) \, \mathrm{d}x = 1.$$

for each Lagrange polynomial, are given by $c_0 = \frac{9}{5}$, $c_1 = \frac{9}{8}$ and $c_2 = \frac{9}{8}$.

Question 4:

(a) What condition on the Butcher array makes a Runge-Kutta scheme implicit?

(b) The differential equation

$$y'(t) = 1 - 2y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{2}\left(e^{-2t} + 1\right)$. From the Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h\sum_{i=1}^{4} b_i f\left(u_k + h\sum_{j=1}^{4} a_{i,j}k_j, t_k + c_i h\right) = u_k + h\sum_{i=1}^{4} b_i k_i$$

and using step size $h = 0.1$, show that to compute u_1, the k coefficients are given by

$$k_1 = -1, \quad k_2 = -0.9, \quad k_3 = -0.91 \quad \text{and} \quad k_4 = -0.818.$$

(c) For an s-stage Runge-Kutta scheme, what condition on the coefficients b_i is necessary for stability?

(d) Find the global truncation error after two steps, i.e. $|y(2h) - u_2|$.

Question 5:

Given the following data:

i	0	1	2
x_i	0	1	3
y_i	2	4	3

Using polynomial interpolation, such as Lagrange or Newton interpolation, what is the value of $y(2)$?

- ○ 14/3
- ○ 4
- ○ 10/3
- ○ 13/3
- ○ 43/9
- ○ 41/10

Question 6:

Given the integral

$$I = \int_{1/4}^{1/5} \frac{1}{3} + \sin(\pi x)\, \mathrm{d}x$$

what is the error of the approximate integral for the Trapezium rule when using five subintervals?

- ○ 2.30e-5
- ○ 2.67e-6
- ○ 8.44e-7
- ○ 2.11e-4
- ○ 8.67e-3
- ○ 8.30e-2

Question 7:

Given the matrix

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

what is the Cholesky matrix L for the matrix A?

○ $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 5 & 1 \end{pmatrix}$ ○ $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & -4 \\ 3 & 8 & 1 \end{pmatrix}$

○ $\begin{pmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 2 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 2 & 1 \\ 2 & 5 \\ 1 & 5 \end{pmatrix}$

○ $\begin{pmatrix} 5 & 3 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$ ○ $\begin{pmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{pmatrix}$

Question 8:

Heun's method for a differential equation $y' = f(y, x)$, can be written in the form:

$$u_{n+1} = u_n + \frac{h}{2}\left(f\left(u_n, x_n\right) + f\left(u^*_{n+1}, x_{n+1}\right)\right) \quad \text{where} \quad u^*_{n+1} = u_n + hf\left(u_n, x_n\right).$$

Show that for the differential equation in $y(x)$ given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y - \ln(x+1), \quad \text{where} \quad y = 1 \quad \text{when} \quad x = 0,$$

that for $h = 0.2$, the value for u_2 is given by:

○ 1.20177 ○ 1.41063

○ 2.40899 ○ 0.98452

○ 1.33333 ○ −0.18431