Constructor University | Spring Semester 2026

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# CTMS-MAT-13: Numerical Methods

Exam: Wednesday 27 May 2026

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**All questions carry equal marks. Only five questions will be marked. Please use the booklet provided, clearly indicating in the inside cover which questions are to be marked.**

**Note that all trigonometric values should be expressed in radians.**

## Question 1:

**(a)** What three operations form the elementary row operations used in Gaussian elimination? [3]

**(b)** Given $a = 3$, find the solution to the linear system $A\vec{x} = \vec{b}$ with

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 3 & -1 & 3 \\ 0 & a & 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

[7]

**(c)** What is a diagonally dominant matrix? [3]

**(d)** For what value of $a$ does the linear system have no solution. [7]

## Question 2:

**(a)** For the function

$$f(x) = x^3 - \sinh(x) + 4x^2 + 6x + 9$$

with initial guesses $x_0 = 7$ and $x_1 = 8$, compute the next three iterations of the secant method

$$x_{k+1} = x_k - f\left(x_k\right) \frac{x_{k-1} - x_k}{f\left(x_{k-1}\right) - f\left(x_k\right)}.$$

[7]

**(b)** State how the secant method is derived from the Newton method. [3]

**(c)** Compute the first three approximations with Newton method with initial guess $x_0 = 7$. [7]

**(d)** Is the rate of convergence of the secant method sublinear, linear or superlinear? [3]

**Question 3:**

**(a)** Define the Kronecker delta, $\delta_{ij}$, as used to define the Lagrange interpolating polynomials. [3]

**(b)** With pairs of data (0,1), (1,2) and (4,2) show that the Lagrange polynomials are

$$\begin{aligned} l_1(x) &= \tfrac{1}{4}(x-1)(x-4) \\ l_2(x) &= \tfrac{1}{3}x(4-x) \\ l_3(x) &= \tfrac{1}{12}x(x-1) \end{aligned}$$

[7]

**(c)** Either using Lagrange or Newton interpolation, show the interpolating polynomial for the data given above is given by

$$p(x) = -\frac{1}{4}x^2 + \frac{5}{4}x + 1$$

[6]

**(d)** Given the error of the $n$-degree polynomial interpolant of a function $f(x)$ through $n+1$ equidistant spaced points is given by

$$f(x) - p_n(x) = \frac{f^{(n+1)}(x)}{(n+1)!}\prod_{i=0}^{n}(x - x_i),$$

describe the Runge phenomena, and how it can be suppressed. [4]

**Question 4:**

**(a)** The differential equation

$$y'(t) = 1 - 2y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{2}\left(e^{-2t} + 1\right)$. From the Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h\sum_{i=1}^{4} b_i f\left(u_k + h\sum_{j=1}^{4} a_{i,j}k_j, t_k + c_i h\right) = u_k + h\sum_{i=1}^{4} b_i k_i$$

and using step size $h = 0.1$, show that to compute $u_1$, the $k$ coefficients are given by

$$k_1 = -1, \quad k_2 = -0.9, \quad k_3 = -0.91 \quad \text{and} \quad k_4 = -0.818.$$

[7]

**(b)** What condition on the Butcher array makes a Runge-Kutta scheme implicit? [3]

**(c)** For an $s$-stage Runge-Kutta scheme, what condition on the coefficients $b_i$ is necessary for stability? [3]

**(d)** Find the global truncation error after two steps, i.e. $|y(2h) - u_2|$. [7]

**Question 5:**

Using Gaussian elimination, or otherwise, what is the solution to the linear system given by $A\boldsymbol{x} = \boldsymbol{b}$ with

$$A = \begin{pmatrix} 1 & 0 & 5 \\ 2 & 2 & -3 \\ 0 & 4 & 4 \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}?$$

- ○ $\left(\frac{1}{6}, \frac{25}{12}, \frac{1}{6}\right)^T$
- ○ $\left(\frac{1}{6}, 1, \frac{1}{6}\right)^T$
- ○ $(1, 1, 1)^T$
- ○ $\left(\frac{1}{2}, \frac{13}{20}, \frac{1}{10}\right)^T$
- ○ $\left(\frac{1}{2}, 1, \frac{1}{2}\right)^T$
- ○ $\left(\frac{25}{12}, \frac{13}{20}, \frac{25}{12}\right)^T$

**Question 6:**

Given the integral

$$I = \int_1^2 \frac{1}{4} + \sin(\pi x)\, \mathrm{d}x$$

what is the error of the approximate integral for the composite Simpson's 1/3 rule

$$S_n = \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n)\right)$$

when using four subintervals?

- ○ 1.06e-4
- ○ 2.67e-6
- ○ 8.44e-7
- ○ 0.38817
- ○ 0.38807
- ○ 8.30e-2

**Question 7:**

The Gauss-Seidel scheme solves the linear equation $A\vec{x} = \vec{b}$ iteratively as

$$(D + L)\vec{x}_{k+1} = -U\vec{x}_k + \vec{b}.$$

where $D$ is the diagonal matrix of $A$ and $L$ is the strictly lower triangular part of $A$. Find the residual, $|\vec{x}^* - \vec{x}_2|$ between the exact solution, $\vec{x}^*$, and the Gauss-Seidel solution after two iterates with initial guess $\vec{x}_0 = (1, -1)^T$, where

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- ○ 0
- ○ 1/5
- ○ 2/5
- ○ 1/2
- ○ 3/4
- ○ 1/4

**Question 8:**

For the vector-valued function

$$f(x,y) = \begin{pmatrix} 4x^2 - 20x + \frac{1}{4}y^2 - 8 \\ \frac{1}{2}xy^2 + 2x - 5y + 8 \end{pmatrix}$$

has the Jacobian matrix $J = J(x,y)$

$$J = \begin{pmatrix} 4(2x-5) & \frac{1}{2}y \\ \frac{1}{2}y^2 - 2 & xy - 5 \end{pmatrix}.$$

Then, let $\vec{u}_n = (x_n, y_n)^T$, and using Newton's method, $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n) f(\vec{u}_n)$, with an initial guess $\vec{u}_0 = (0,0)^T$, show that the first iteration of Newton's method is

- ○ $(0.3, 2.33)^T$
- ○ $(0.0, -1.44)^T$
- ○ $(-0.1, -2.33)^T$
- ○ $(-0.4, 1.76)^T$
- ○ $(-0.1, 3.22)^T$
- ○ $(0.2, 1.22)^T$