Constructor University **Spring Semester 2024**

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JTMS-MAT-13: Numerical Methods

Sample Exam Questions

Question 1:
Consider the linear system of equations $Ax = b$ with

$$\begin{aligned} 5x_1 + 2x_2 - 2x_3 &= 10 \\ x_1 + 6x_2 - 2x_3 &= -6 \\ x_2 + 4x_3 &= 8 \end{aligned}$$

(a) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next iterate $\vec{x}_1$ of the Jacobi iteration.

(b) Write down a sufficient condition on a linear system to yield convergence of the Jacobi iteration.

(c) Given the two equations $2x_1 + 5x_2 = 16$ and $3x_1 + x_2 = 11$, formulate this as a system of equations $Ax = b$ in such an order that convergence is guaranteed and, using the starting point $\vec{x}_0 = \vec{0}$, compute the next iterate $\vec{x}_1$ of the Gauss-Seidel iteration.

Question 2:
Take the function

$$f(x) = x^2 - 4x + 1.$$

We want to find root of this function.

(a) Taking the starting point $x_0 = 2$. Why can we not apply the Newton method?

(b) Derive the secant method from the Newton method.

(c) Using the points $x_0 = 2$ and $x_1 = 3$ carry out one step of the secant method.

(d) Show that we can formally apply the Newton method with the starting point $x_0 = 3$ and do one iteration.

(e) By calculating the actual root(s) of the function, which of the two schemes of **(c)** and **(d)** has the lower error with respect to the closest root? Briefly describe if you would expect that from their convergence behaviour.

Question 3:
Consider the interpolation problem

i	0	1	2
x_i	0	π	3π
p_i	1	-1	2

We try to fit the function $p(x) = a \sin(x/2) + b \cos(2x)$.

(a) Solve the resulting overdetermined system to find the free coefficients a and b by either using the 3×2 collocation matrix and its transpose or by writing down the system of normal equations. Both methods are based on the minimization of the sum of squared errors.

(b) Briefly describe the terms over-determined and under-determined in the context of linear systems of equations.

Question 4:
For the ordinary differential equation

$$y'(t) = -y(t) + t^2, \quad y(0) = 1.$$

(**a**) Perform one step each with the forward Euler and backward Euler schemes, with the step $h = 1/2$ starting at $y(0)$.

(**b**) Perform one step of the so called Heun's third-order method which follows the Butcher array

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{3} & \frac{1}{3} & & \\ \frac{2}{3} & & \frac{2}{3} & \\ \hline & \frac{1}{4} & & \frac{3}{4} \end{array}$$

with the step $h = 1/2$ starting at $y(0)$.

(**c**) Show that $y(t) = t^2 - 2t - e^{-t} + 2$ is a solution to the ODE with the initial condition $y(0) = 1$ and compute the errors for (**a**) and (**b**).

(**d**) Briefly explain the terms explicit and implicit in this context. Which of the above methods are implicit, which are explicit? What is the advantage of implicit methods compared to explicit methods, what is their disadvantage?

Question 5:
Given the following integral

$$\int_0^{2\pi/\sqrt{3}} \sin(t)\, f(t)\, \mathrm{d}t$$

with some function $f(t)$.

(**a**) Derive the Legendre polynomial with $n = 1$ and its two Gaussian nodes.

(**b**) Evaluate the integral with $f(t) = \cos(t)$, using Gaussian quadrature based on the Gaussian nodes found in (**a**).

(**c**) Show that $\sin^2(t)/2$ is the indefinite integral. Using this, compute the exact solution. How large is the error of Gaussian quadrature?

(**d**) What specific integration rule does this kind of Gaussian quadrature correspond to (neglecting the exact choice of nodes)?

Question 6:
Given the following interpolation problem with points p_i at nodes u_i

i	0	1	2
x_i	0	3	5
p_i	1	-2	6

Consider the Newton polynomials

$$p_i(u) = \prod_{j=0}^{i-1} (u - u_i) \quad \text{with} \quad p_0(u) = 1.$$

(**a**) Derive the collocation matrix for the given interpolation problem.

(**b**) Write down the collocation matrix for the case of Lagrange interpolation.

(**c**) Use the collocation matrix from (**a**) to compute the interpolating polynomial $p_2(u)$ for the first two points only (i.e. $i = 0$ and $i = 1$).