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JTMS-MAT-13: Numerical Methods

Exam: Saturday 24 August 2024

All questions carry equal marks. Answer 5 questions only. Please use the booklet provided, clearly stating which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

(a) State a condition which means a square matrix will not be invertible.

(b) Given the matrix

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix},$$

use Gaussian elimination to show the row echelon form of the matrix A as

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) By applying Gaussian elimination, or any other method, show the solution to the linear equation $A\vec{x} = \vec{b}$, where $\vec{b}$ is given by

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{is} \quad \vec{x} = \begin{pmatrix} 110.25 \\ -24 \\ 9.5 \end{pmatrix}.$$

(d) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

Question 2

(a) Find the Jacobian matrix $J = J(x, y)$ for the vector-valued function

$$f(x,y) = \begin{pmatrix} 4x^2 - 20x + \frac{1}{4}y^2 - 8 \\ \frac{1}{2}xy^2 + 2x - 5y + 8 \end{pmatrix}.$$

(b) Show the inverse of the Jacobian matrix is

$$J^{-1} = \frac{1}{|J|}\begin{pmatrix} xy - 5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2 - 2 & 4(2x-5) \end{pmatrix} \quad \text{where} \quad |J| = 8x^2y - 40x - 20xy + 100 - \frac{1}{4}y^3 - y.$$

(c) Let $\vec{u}_n = (x_n, y_n)^T$. Then, using Newton's method, $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n)\, f(\vec{u}_n)$, with an initial guess $\vec{u}_0 = (0,0)^T$, show that the first iteration of Newton's method is $(-0.4, 1.44)^T$.

Question 3: Consider the integral

$$I = \int_1^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{\mathrm{d}x}{x} = \ln(2) = 0.6931471805599453$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-2} f(x_i)\right)$$

where $h = (b-a)/n$ and n is the number of intervals, show that the approximations to the integral for $n = 2^k$ where $k = 0$, 1 and 2 are

k	n	I_n
0	1	0.75
1	2	0.70833333
2	4	0.69702381

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \cdots$$

for some constants a, and considering the difference between the errors of the Trapezium rule for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3}\left(4R_k^0 - R_{k-1}^0\right)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

Question 4:

(a) For the numerical solutions of ordinary differential equations, $y' = f(y,t)$, what is an explicit method?

(b) Using a Taylor expansion, show the forward Euler scheme for a first-order differential equation is

$$y_{n+1} = y_n + hf(y_n, t_n)$$

where y_n denotes the solution at $t_n = t_0 + nh$ for a time step h and initial condition $y(t_0)$.

(c) For the second-order linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1$$

show, by considering the backward difference approximation $\nabla\vec{u}(t_{n+1}) \approx (\vec{u}_{n+1} - \vec{u}_n)/h$, where $\vec{u} = (y, y')$ and $\vec{u}_n$ denotes the solution at $t_n = t_0 + nh$, that the backward Euler scheme yields

$$\vec{u}_{n+1} = \frac{1}{1 + 4h - h^2}\begin{pmatrix} 1+4h & -h \\ -h & 1 \end{pmatrix}\vec{u}_n.$$

(d) Compute the first two steps of the backward Euler scheme for the system given in **(c)**, i.e. $\vec{u}_0 = (1,1)$ with $h = 0.1$.

Question 5: Integrals can be numerically evaluated using the composite Simpson's 1/3 rule, which is given by

$$I = \int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n)\right)$$

where $x_i = a + ih$, with $h = (b-a)/n$ for $i = 0, \ldots, n$, where n, the number of subintervals, is even. Given the integral

$$I = \int_0^1 \frac{1}{3} + \cos(\pi x)\,\mathrm{d}x$$

what is the difference between the exact and the approximate integral using Simpson's rule with six subintervals?

- ○ 1/2
- ○ 1/6
- ○ 0
- ○ 1/10
- ○ 1/3
- ○ 1/7

Question 6: Given the following data:

i	0	1	2
x_i	0	2	4
y_i	2	1	2

Newton interpolation constructs a interpolating polynomial $p(x)$, using the formula $p = \sum_{i=0}^{n} \alpha_i n_i(x)$, where the basis polynomials are defined as

$$n_0(x) = 1, \quad n_i(x) = (x - x_0)(x - x_1)\cdots(x - x_{i-1}) \quad \text{for} \quad i \geq 1$$

where $p(x_i) = y_i$. Using Newton interpolation, which are the correct collocation matrix Φ and weighting vector $\vec{\alpha}$ such that $\Phi\vec{\alpha} = \vec{y}$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)^T$ and $\vec{y} = (y_0, y_1, y_2)^T$.

- ○ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ -1/4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -1/2 \\ -1/3 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Question 7: Using the Jacobi scheme, $\vec{x}_{n+1} = \left(I - D^{-1}A\right)\vec{x}_n + D^{-1}b$, where D is the diagonal matrix of A, what is the second iterate, $\vec{x}_2$, of the solution to the system $Ax = b$ with

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

- ○ $\begin{pmatrix} 0.456 \\ 0.7421 \\ -0.8123 \end{pmatrix}$
- ○ $\begin{pmatrix} 0.6694 \\ 0.2306 \\ 0.5183 \end{pmatrix}$
- ○ $\begin{pmatrix} 1.7326 \\ -0.6034 \\ 0.6777 \end{pmatrix}$
- ○ $\begin{pmatrix} 0.6789 \\ 0.2481 \\ 0.1111 \end{pmatrix}$
- ○ $\begin{pmatrix} 0.0013 \\ 0.0299 \\ 3.5217 \end{pmatrix}$
- ○ $\begin{pmatrix} 2.9876 \\ 0.1234 \\ 2.3540 \end{pmatrix}$

Question 8: The differential equation

$$y'(t) = 1 - 3y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{3}\left(2e^{-3t} + 1\right)$. From the explicit Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}$$

where

$$u_{k+1} = u_k + h \sum_{i=1}^{4} b_i f\left(u_k + h \sum_{j=1}^{4} a_{i,j} k_j, t_k + c_i h\right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 0.002
- ○ 0.058
- ○ 0.370
- ○ 0.115
- ○ 0.965
- ○ 0.262