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JTMS-MAT-13: Numerical Methods

Exam & Solutions: Saturday 24 August 2024

All questions carry equal marks. Answer 5 questions only. Please use the booklet provided, clearly stating which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

(a) State a condition which means a square matrix will not be invertible.

(b) Given the matrix

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix},$$

use Gaussian elimination to show the row echelon form of the matrix A is given by

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) By applying Gaussian elimination, or any other method, show the solution to the linear equation $A\vec{x} = \vec{b}$, where $\vec{b}$ is given by

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{is} \quad \vec{x} = \begin{pmatrix} 110.25 \\ -24 \\ 9.5 \end{pmatrix}.$$

(d) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

(a) A matrix is not invertible if the determinant is zero. Equivalent conditions, such as if rank is not full, or rows/columns are not linearly independent are also acceptable.

(b) To express the matrix in row echelon form, start by eliminating the element in the first column of the second row by subtracting 3 times the first row from the second row:

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 - 3 \cdot 4 & 40 - 3 \cdot 12 & -38 - 3 \cdot (-16) \\ -16 & -38 & 90 \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ -16 & -38 & 90 \end{pmatrix}.$$

Next, eliminate the element in the first column of the third row. To do this, add 4 times the first row to the third row

$$\begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ -16 + 4 \cdot 4 & -38 + 4 \cdot 12 & 90 + 4 \cdot (-16) \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 10 & 26 \end{pmatrix}.$$

Finally, eliminate the element in the second column of the third row by subtracting 2.5 times the second row from the third row

$$\begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 10 - 2.5 \cdot 4 & 26 - 2.5 \cdot 10 \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

The final upper triangular form of the matrix A is:

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Applying the elementary row operations to the righthand side vectors,

$$\left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 12 & 40 & -38 & 2 \\ -16 & -38 & 90 & 3 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ -16 & -38 & 90 & 3 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ 0 & 10 & 26 & 7 \end{array}\right)$$

and

$$\left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ 0 & 10 & 26 & 7 \end{array}\right) \mapsto \left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ 0 & 0 & 1 & 9.5 \end{array}\right)$$

Thus if the unknown vector $\vec{x} = (x_1, x_2, x_3)$, then $x_3 = 9.5$, so that $4x_2 + 95 = -1 \Rightarrow x_2 = -24$ and finally $4x_1 + 12 \times 24 + 9.5 = 1 \Rightarrow x_1 = 110.25$.

(d) The number of operations is proportional to the cube of the number of rows, i.e. $\mathcal{O}\left(n^3\right)$.

Question 2

(a) Find the Jacobian matrix for the vector-valued function

$$f(x,y) = \begin{pmatrix} 4x^2 - 20x + \frac{1}{4}y^2 - 8 \\ \frac{1}{2}xy^2 + 2x - 5y + 8 \end{pmatrix}.$$

(b) Show the inverse of the Jacobian matrix is

$$J^{-1} = \frac{1}{|J|}\begin{pmatrix} xy - 5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2 - 2 & 4(2x-5) \end{pmatrix}, \quad \text{where} \quad |J| = 8x^2y - 20xy - 40x + 100 - \frac{1}{4}y^3 - y.$$

(c) Let $\vec{u}_n = (x_n, y_n)^T$. Then, using $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n)\, f(\vec{u}_n)$, with an initial guess $\vec{u}_0 = (0,0)^T$, show that the first iteration of Newton's method yields $(-0.4, 1.44)^T$.

(a) Compute the partial derivatives:

$$\begin{aligned}
\frac{\partial f_1}{\partial x} &= \frac{\partial}{\partial x}\left(4x^2 - 20x + \frac{1}{4}y^2 - 8\right) \\
&= 8x - 20 = 4(2x-5) \\
\frac{\partial f_1}{\partial y} &= \frac{\partial}{\partial y}\left(4x^2 - 20x + \frac{1}{4}y^2 - 8\right) \\
&= \frac{1}{2}y \\
\frac{\partial f_2}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{1}{2}xy^2 + 2x - 5y + 8\right) \\
&= \frac{1}{2}y^2 + 2 \\
\frac{\partial f_2}{\partial y} &= \frac{\partial}{\partial x}\left(\frac{1}{2}xy^2 + 2x - 5y + 8\right) \\
&= xy - 5
\end{aligned}$$

Thus, the Jacobian matrix J is:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 4(2x-5) & \frac{1}{2}y \\ \frac{1}{2}y^2 + 2 & yx - 5 \end{pmatrix}$$

(b) For the inverse, first compute the determinant:

$$\begin{aligned}
\det(J) &= 4(2x-5)(xy-5) - \frac{1}{2}y\left(\frac{1}{2}y^2 + 2\right) \\
&= 4(2x^2y - 5xy - 10x + 25) - \frac{1}{4}y^3 - y \\
&= 8x^2y - 20xy - 40x + 100 - \frac{1}{4}y^3 - y
\end{aligned}$$

The formula for the inverse of a 2×2 matrix is given by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \frac{1}{\det(A)}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{when} \quad \det(A) \neq 0.$$

Thus,

$$J^{-1} = \frac{1}{|J|}\begin{pmatrix} xy-5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2-2 & 4(2x-5) \end{pmatrix}, \quad \text{where} \quad |J| = 8x^2y - 20xy - 40x + 100 - \frac{1}{4}y^3 - y.$$

[(**c**) Evaluating the inverse of the Jacobian at the initial guess yields

$$J^{-1}\left(\vec{u}_0\right) = \frac{1}{100}\begin{pmatrix} -5 & 0 \\ -2 & -20 \end{pmatrix}.$$

With $\vec{u}_0 = (0,0)^T$, then

$$f\left(u_0\right) = \begin{pmatrix} -8 \\ +8 \end{pmatrix},$$

so

$$\vec{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{100}\begin{pmatrix} -5 & 0 \\ -2 & -20 \end{pmatrix}\begin{pmatrix} -8 \\ +8 \end{pmatrix} = \frac{1}{100}\begin{pmatrix} -40 \\ 144 \end{pmatrix}.$$

Question 3: Consider the integral

$$I = \int_1^2 f(x)\,\mathrm{d}x = \int_1^2 \frac{\mathrm{d}x}{x} = \ln(2) = 0.6931471805599453$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2}\left(f(x_0) + f(x_n) + 2\sum_{i=1}^{n-2} f(x_i)\right)$$

where $h = (b-a)/n$ and n is the number of intervals, show that the approximations to the integral for $n = 2^k$ where $k = 0$, 1 and 2 are

k	n	I_n
0	1	0.75
1	2	0.70833333
2	4	0.69702381

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \cdots$$

for some constants a, and considering the difference between the errors of the Trapezium rule for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3}\left(4R_k^0 - R_{k-1}^0\right)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

(a) For the first value $h = 1$,

$$R_0^0 = \frac{1}{2}\left(1 + \frac{1}{2}\right) = 0.75.$$

For the second value, $k = 1$, $n = 2$,

$$h = \frac{1-0}{2} = 0.5, \qquad x_0 = 1, \quad x_1 = 1.5 \quad \text{and} \quad x_2 = 2.$$

Using the function $f(x) = \dfrac{1}{x}$

$$f(1) = \frac{1}{1} = 1$$
$$f(1.5) = \frac{1}{1.5} = 2/3$$
$$f(2) = \frac{1}{2} = 0.5$$

Substituting these values into the formula:

$$I_1 = \frac{0.5}{2}\left(1 + 0.5 + 2\cdot 2/3\right) = 0.25\left(1.5 + 4/3\right) = 0.70833$$

For the next value, $k = 2$, $n = 2^2 = 4$, $h = 1/4$

$$
\begin{aligned}
f(1) &= 1 \\
f(1.25) &= \frac{1}{1.25} = \frac{1}{5/4} = \frac{4}{5} = 0.8 \\
f(1.5) &= 2/3 \\
f(1.75) &= \frac{1}{1.75} = \frac{1}{7/4} = \frac{4}{7} = 0.5714285714285714 \\
f(2) &= 0.5
\end{aligned}
$$

Substituting these values into the formula:

$$I_2 = \frac{0.25}{2}\left(1 + 0.5 + 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right)\right) = 0.69702380952$$

(b) Consider the expansion for I_{i-2} and I_{i-1}, noting that I_{i-1} has twice the steps size.

$$\int_a^b f(x)\,\mathrm{d}x = R_{i-1}^0 + a_2 h^2 + a_4 h^4 + \ldots$$

and

$$\int_a^b f(x)\,\mathrm{d}x = R_i^0 + a_2\left(\frac{h}{2}\right)^2 + a_4\left(\frac{h}{2}\right)^4 + \ldots$$

then an $\mathcal{O}\left(h^4\right)$ approximation is made by multiplying I_{i-1} by four and subtracting I_{i-2}. Let this be denoted by R_1^k. This gives a value which approximates three times the integral, thus the new formula is, as required

$$R_k^1 = \frac{1}{3}\left(4R_k^0 - R_{k-1}^0\right).$$

(c) By the formula,

$$R_2^1 = \frac{1}{3}\left(4R_2^0 - R_1^0\right) = \frac{1}{3}\left(4 \cdot 0.69702380952 - 0.70833333333\right) = 0.69325396825.$$

Question 4:

(a) For the numerical solutions of ordinary differential equations, $y' = f(y,t)$, what is an explicit method?

(b) Using a Taylor expansion, show the forward Euler scheme for a first-order differential equation is

$$y_{n+1} = y_n + hf(y_n, t_n)$$

where y_n denotes the solution at $t_n = t_0 + nh$ for a time step h and initial condition $y(t_0)$.

(c) For the second-order linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1$$

show, by considering the backward difference approximation $\nabla\vec{u}(t_{n+1}) \approx (\vec{u}_{n+1} - \vec{u}_n)/h$, where $\vec{u} = (y, y')$ and $\vec{u}_n$ denotes the solution at $t_n = t_0 + nh$, that the backward Euler scheme yields

$$\vec{u}_{n+1} = \frac{1}{1 + 4h - h^2}\begin{pmatrix} 1+4h & -h \\ -h & 1 \end{pmatrix}\vec{u}_n.$$

(d) Compute the first two steps of the backward Euler scheme for the system given in **(c)** with $h = 0.1$.

(a) The function to be solved only involves the current and past values of the unknown function.

(b) The Taylor expansion is given by

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2!}y''(t_n) + \cdots$$

where $t_{n+1} = t_n + h$ and h is the step size. For the forward Euler scheme, we will truncate the series after the first-order term

$$y(t_{n+1}) \approx y(t_n) + hy'(t_n)$$

and as $y'(t_n) = f(t_n)$, so

$$y_{n+1} = y_n + hf(t_n, y_n).$$

(c) Rearranging the expression, yields $\vec{u}_{n+1} = \vec{u}_n + h\nabla(\vec{u}_{n+1})$ The backwards Euler scheme is given by $u_{n+1} = u_n + hf(u_{n+1})$, i.e. $\vec{v}_{n+1} = \vec{v}_n + hA\vec{v}_{n+1}$ as the system is linear and can be written as $\nabla\nabla v = A\vec{v}$. Thus, $(I - hA)\vec{v}_{n+1} = \vec{v}_n$, so that $\vec{v}_{n+1} = (I - hA)^{-1}\vec{v}_n$. The inverse of the matrix for the linear system is then given by

$$\begin{aligned}(I - hA)^{-1} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - h\begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}\right)^{-1} \\ &= \begin{pmatrix} 1 & -h \\ -h & 1+4h \end{pmatrix}^{-1} \\ &= \frac{1}{1+4h-h^2}\begin{pmatrix} 1+4h & -h \\ -h & 1 \end{pmatrix}.\end{aligned}$$

(d) With step size $h = 0.1$, then two steps, $\vec{u}_1$ and $\vec{u}_2$, must be computed from the initial data $\vec{u}_0 = (0, 1)$.

Using the matrix derived in the formula given, the solution at $t = 0.1$ is given by

$$
\begin{aligned}
\vec{u}_1 &= \frac{1}{1 + 4 \times 0.1 - 0.1^2} \begin{pmatrix} 1 + 4 \times 0.1 & -0.1 \\ -0.1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{100}{139} \begin{pmatrix} 1.4 & -0.1 \\ -0.1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{100}{139} \begin{pmatrix} 140 & -10 \\ -10 & 100 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= \frac{100}{139} \begin{pmatrix} 140 & -10 \\ -10 & 100 \end{pmatrix} \\
&= \begin{pmatrix} -10/139 \\ 100/139 \end{pmatrix} \\
&= \begin{pmatrix} -0.071942 \\ 0.719424 \end{pmatrix}.
\end{aligned}
$$

The second step is then given by

$$
\begin{aligned}
\vec{u}_2 &= \frac{1}{139} \begin{pmatrix} 140 & -10 \\ -10 & 100 \end{pmatrix} \begin{pmatrix} -10/139 \\ 100/139 \end{pmatrix} \\
&= \begin{pmatrix} 1.007194 & -0.07194244604316546 \\ -0.07194244604316546 & 0.7194244604316546 \end{pmatrix} \begin{pmatrix} -0.071942 \\ 0.719424 \end{pmatrix} \\
&= \begin{pmatrix} -0.12421717 \\ 0.52274727 \end{pmatrix}.
\end{aligned}
$$

Question 5: Integrals can be numerically evaluated using the composite Simpson's 1/3 rule, which is given by

$$I = \int_a^b f(x)\,\mathrm{d}x \approx \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{n/2} f(x_{2i-1}) + 2\sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n)\right)$$

where $x_i = a + ih$, with $h = (b-a)/n$ for $i = 0, \ldots, n$, where n, the number of subintervals, is even. Given the integral

$$I = \int_0^1 \frac{1}{3} + \cos(\pi x)\,\mathrm{d}x$$

what is the difference between the exact and the approximate integral using Simpson's rule with six subintervals?

- ○ 1/2
- ✗ 0
- ○ 1/3
- ○ 1/6
- ○ 1/10
- ○ 1/7

The integral is given by:

$$\begin{aligned} I = \int_0^1 \frac{1}{3} + \cos(\pi x)\,\mathrm{d}x &= \left[\frac{x}{3} - \frac{\sin(\pi x)}{\pi}\right]_0^1 \\ &= \left[\frac{x}{3}\right]_0^1 - \left[\frac{\sin(\pi x)}{\pi}\right]_0^1 \\ &= \frac{1}{3} - 0 - \frac{1}{\pi}(\sin(0) - \sin(\pi)) = \frac{1}{3} \end{aligned}$$

For the approximation, with $a = 0$, $b = 1$, and $n = 6$:

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}.$$

The points x_i are:

$$x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{2}{6} = \frac{1}{3}, \quad x_3 = \frac{1}{2}, \quad x_4 = \frac{2}{3}, \quad x_5 = \frac{5}{6} \quad \text{and} \quad x_6 = 1.$$

The function values $f(x_i) = \frac{1}{3} + \cos(\pi x_i)$ are:

$$\begin{aligned} f(x_0) &= \frac{1}{3} + \cos(0) = \frac{1}{3} + 1 = \frac{4}{3}, \\ f(x_1) &= \frac{1}{3} + \cos\left(\frac{\pi}{6}\right) = \frac{1}{3} + \frac{\sqrt{3}}{2}, \\ f(x_2) &= \frac{1}{3} + \cos\left(\frac{\pi}{3}\right) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}, \\ f(x_3) &= \frac{1}{3} + \cos\left(\frac{\pi}{2}\right) = \frac{1}{3} + 0 = \frac{1}{3}, \\ f(x_4) &= \frac{1}{3} + \cos\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}, \\ f(x_5) &= \frac{1}{3} + \cos\left(\frac{5\pi}{6}\right) = \frac{1}{3} - \frac{\sqrt{3}}{2}, \\ f(x_6) &= \frac{1}{3} + \cos(\pi) = \frac{1}{3} - 1 = -\frac{2}{3}. \end{aligned}$$

Now apply Simpson's rule:

$$\begin{aligned} I &\approx \frac{\Delta x}{3}\left[f(x_0) + 4(f(x_1) + f(x_3) + f(x_5)) + 2(f(x_2) + f(x_4)) + f(x_6)\right]. \\ &= \frac{1/6}{3}\left[\frac{4}{3} + 4\left(\frac{1}{3} + \frac{\sqrt{3}}{2} + + \frac{1}{3} + \frac{1}{3} - \frac{\sqrt{3}}{2}\right) + 2\left(\frac{5}{6} - \frac{1}{6}\right) - \frac{2}{3}\right]. \end{aligned}$$

Simplify the sums:

$$\begin{aligned}
I &\approx \frac{1}{18}\left[\frac{4}{3} + 4\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right) + 2\left(\frac{4}{6}\right) + -\frac{2}{3}\right]. \\
&= \frac{1}{18}\left[\frac{4}{3} + 4 + 2\left(\frac{2}{3}\right) + -\frac{2}{3}\right] \\
&= \frac{1}{18}\left[\frac{4}{3} + \frac{12}{3} + \frac{4}{3} - \frac{2}{3}\right] \\
&= \frac{1}{18}\left[\frac{4 + 12 + 4 - 2}{3}\right] \\
&= \frac{1}{18}\left[\frac{18}{3}\right] \\
&= \frac{1}{3}.
\end{aligned}$$

Thus, the difference between the approximation and the exact value is

$$\left|\frac{1}{3} - \frac{1}{3}\right| = 0$$

Question 6: Given the following data:

i	0	1	2
x_i	0	2	4
y_i	2	1	2

Newton interpolation constructs a interpolating polynomial $p(x)$, using the formula $p = \sum_{i=0}^{n} \alpha_i n_i(x)$, where the basis polynomials are defined as

$$n_0(x) = 1, \quad n_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1})$$

where $p(x_i) = y_i$. Using Newton interpolation, which are the correct collocation matrix Φ and weighting vector $\vec{\alpha}$ such that $\Phi\vec{\alpha} = \vec{y}$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$ and $\vec{y} = (y_0, y_1, y_2)$.

- ☒ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ -1/4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -1/2 \\ -1/3 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The Newton interpolating polynomial can be written in the form:

$$p(x) = \alpha_0 + \alpha_1 (x - x_0) + \alpha_2 (x - x_0)(x - x_1)$$

For the given data points (x_0, y_0), (x_1, y_1) and (x_2, y_2), the collocation matrix Φ is constructed by evaluating the basis functions of the Newton polynomial at each x_i, where the basis functions are given by $n_0 = 1$, $n_1 = x - x_0$ and $n_2 = (x - x_0)(x - x_1)$.

- For $x_0 = 0$, $n_0(x_0) = 1$, $n_1(x_0) = 0$ and $n_2(x_0) = 0$.
- For $x_1 = 2$, $n_0(x_1) = 1$, $n_1(x_1) = 2$ and $n_2(x_1) = 0$.
- For $x_2 = 4$, $n_0(x_2) = 1$, $n_1(x_2) = 4$ and $n_2(x_2) = 8$.

The system $\Phi\vec{\alpha} = \vec{y}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Therefore, the correct collocation matrix Φ is:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}.$$

and

$$\vec{\alpha} = \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}.$$

So that

$$p = 2 - \frac{x}{2} + \frac{x}{4}(x - 2)$$

Question 7: Using the Jacobi scheme, $\vec{x}_{n+1} = \left(I - D^{-1}A\right)\vec{x}_n + D^{-1}b$, where D is the diagonal matrix of A, what is the second iterate, $\vec{x}_2$, of the solution to the system $Ax = b$ with

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

○ $\begin{pmatrix} 0.456 \\ 0.7421 \\ -0.8123 \end{pmatrix}$ ☒ $\begin{pmatrix} 0.6694 \\ 0.2306 \\ 0.5183 \end{pmatrix}$

○ $\begin{pmatrix} 1.7326 \\ -0.6034 \\ 0.6777 \end{pmatrix}$ ○ $\begin{pmatrix} 0.6789 \\ 0.2481 \\ 0.1111 \end{pmatrix}$

○ $\begin{pmatrix} 0.0013 \\ 0.0299 \\ 3.5217 \end{pmatrix}$ ○ $\begin{pmatrix} 2.9876 \\ 0.1234 \\ 2.3540 \end{pmatrix}$

First, extract the diagonal matrix D from A:

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 90 \end{pmatrix}.$$

Next, compute D^{-1}:

$$D^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{40} & 0 \\ 0 & 0 & \frac{1}{90} \end{pmatrix}.$$

Now, compute the matrix $I - D^{-1}A$:

$$D^{-1}A = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{40} & 0 \\ 0 & 0 & \frac{1}{90} \end{pmatrix} \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 \\ 0.3 & 1 & -0.95 \\ -\frac{8}{45} & -\frac{19}{45} & 1 \end{pmatrix}.$$

So

$$I - D^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 & -4 \\ 0.3 & 1 & -0.95 \\ -\frac{8}{45} & -\frac{19}{45} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix}.$$

Next, compute $D^{-1}\vec{b}$:

$$D^{-1}\vec{b} = \begin{pmatrix} 1/4 & 0 & 0 \\ 0 & 1/40 & 0 \\ 0 & 0 & 1/90 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix}.$$

Now, iteratively compute $\vec{x}_1$ and then $\vec{x}_2$

$$\vec{x}_1 = \left(I - D^{-1}A\right)\vec{x}_0 + D^{-1}\vec{b} = \begin{pmatrix} 0 & -3 & 4 \\ -\frac{3}{10} & 0 & \frac{19}{20} \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix}.$$

First, compute the matrix-vector product:

$$\begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 - 3 + 4 \\ -0.3 + 0 + 0.95 \\ \frac{8}{45} + \frac{19}{45} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.65 \\ \frac{27}{45} \end{pmatrix}.$$

So

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0.65 \\ 0.6 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.675 \\ 0.6111 \end{pmatrix}.$$

The $\vec{x}_2$ is given by

$$\vec{x}_2 = \left(I - D^{-1}A\right)\vec{x}_1 + D^{-1}\vec{b} = \begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix}\begin{pmatrix} 1.25 \\ 0.675 \\ 0.6111 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix}.$$

Compute the matrix-vector product:

$$\begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix}\begin{pmatrix} 1.25 \\ 0.675 \\ 0.6111 \end{pmatrix} = \begin{pmatrix} 0 - 2.025 + 2.4444 \\ -0.375 + 0 + 0.580545 \\ \frac{10}{45} + \frac{12.825}{45} \end{pmatrix} = \begin{pmatrix} 0.4194 \\ 0.205545 \\ 0.5072 \end{pmatrix}.$$

So:

$$\vec{x}_2 = \begin{pmatrix} 0.4194 \\ 0.205545 \\ 0.5072 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix} = \begin{pmatrix} 0.6694 \\ 0.230545 \\ 0.5183 \end{pmatrix}.$$

Therefore, the second iterate $\vec{x}_2$ is approximately:

$$\vec{x}_2 = \begin{pmatrix} 0.6694 \\ 0.2305 \\ 0.5183 \end{pmatrix}.$$

Question 8: The differential equation

$$y'(t) = 1 - 3y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{3}\left(2e^{-3t} + 1\right)$. From the explicit Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h \sum_{i=1}^{4} b_i f\left(u_k + h \sum_{j=1}^{4} a_{i,j} k_j, t_k + c_i h\right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 0.002
- ○ 0.058
- ○ 0.370
- ○ 0.115
- ○ 0.965
- ☒ 0.262

Using the formula given, the Butcher array yields a Runge-Kutta scheme of the form:

$$\begin{aligned} k_1 &= hf\left(u_n, t_n\right) \\ k_2 &= hf\left(u_n + hk_1/2, t_n + h/2\right) \\ k_3 &= hf\left(u_n + hk_2/2, t_n + h/2\right) \\ k_4 &= hf\left(u_n + hk_3, t_n + h\right) \end{aligned}$$

for the function which is not dependent on time

$$f(u_n) = 1 - 3u_n$$

which generates the next value via

$$u_{n+1} = u_n + (h/6)\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

Given the initial condition and the step size, to compute two time steps means to compute approximations to $y(0.1)$ and $y(0.2)$. For the first time step, with $u_0 = 1.0$ and $h = 0.1$,

$$k_1 = hf\left(u_0\right) = 0.1\left(1 - 3\right) = -0.2.$$

Thus the first step can be computed using the values

$$k_1 = -0.2, \quad k_2 = -0.197, \quad k_3 = -0.197045 \quad \text{and} \quad k_4 - 0.19408865.$$

So that

$$u_1 = 1.0 + (0.1/6)(-0.2 - 2\cdot 0.197 - 2\cdot 0.197045 - 0.19408865) = 0.9802970225.$$

The second evaluation yields

$$k_1 = -0.1940891, \quad k_2 = -0.1911778, \quad k_3 - 0.1912214 \quad \text{and} \quad -0.1883525$$

Thus $u_2 = 0.9611764$.

The exact values are given by $y(0.1) = 0.8272122$ and $y(0.2) = 0.6992078$, thus the difference at $t = 0.2$ is

$$|y(0.2) - u_2| = 0.26196860.$$