Math6502 test 16$^{\text{th}}$ January 2009

This test will last exactly 90 minutes.

Attempt all questions.

This is an open book test: lecture notes and text books are permitted in this exam. The use of an electronic calculator is permitted in this test.

1. (*i*) Write down the definition for the inverse of a matrix $\underline{\underline{B}}$. [2]

   (*ii*) Consider the matrix: [8]

$$\underline{\underline{C}} = \begin{pmatrix} 1 & 0 & 0 \\ 1+\sqrt{2} & 2 & \sqrt{2} \\ 0 & \sqrt{2} & x \end{pmatrix}.$$

   For what values of x does the inverse of the matrix $\underline{\underline{C}}$ not exist?

   (*iii*) Consider the following system of linear equations: [5]

$$\begin{aligned} 3(x_1 + x_2) &= 1 - x_3, \\ x_1 + x_3 - 3 &= x_2, \\ 2x_2 &= 2 - x_3. \end{aligned}$$

   Write this system of equations in the form $\underline{\underline{A}}\,\underline{\mathbf{x}} = \underline{\mathbf{b}}$, where $\underline{\mathbf{x}} = (x_1, x_2, x_3)^T$ and $\underline{\underline{A}}$ is a constant (3×3)-matrix and $\underline{\mathbf{b}}$ is a constant three-dimensional vector.

   (*iv*) Either by inverting $\underline{\underline{A}}$, or using Gaussian elimination to express $\underline{\underline{A}}$ in row echelon form and using back substitution, find x_1, x_2 and x_3 and then verify your answer. [10]

2. (*i*) Find the first three nonzero terms of the Maclaurin series for [5]

$$f(x) = \sin(n\pi x).$$

   Is the function odd, even or neither?

   (*ii*) Show that [7]

$$\int_0^{1/2} x \sin(n\pi x)\,\mathrm{d}x = \frac{(-1)^{n+1}}{\pi^2 n^2}.$$

   (*iii*) Sketch the periodic continuation, with period 1, of the function [5]

$$f(x) = x \quad \text{for} \quad -1/2 < x \leq 1/2$$

   (*iv*) Find the Fourier series for $f(x)$. [8]

3. A one-dimensional iron bar of unit length is positioned between $x = 0$ and $x = 1$. The temperature of the bar is given by the function $f(x,t)$. The initial temperature profile at time $t = 0$ is given by

$$f(x,0) = \begin{cases} x & 0 < x \leq 1/2, \\ (1-x) & 1/2 < x \leq 1. \end{cases}$$

Additionally the temperature of the bar is held at zero at both ends for all time.

(*i*) Write down the one-dimensional heat equation in Cartesian coordinates. [2]

(*ii*) Using separation of variables derive the general solution of the one-dimensional heat equation to show that the general solution takes the form [10]

$$f(x,t) = C + Dx + \sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t}\left(a_n \cos\lambda_n x + b_n \sin\lambda_n x\right) + \sum_{n=1}^{\infty} e^{\kappa\mu_n^2 t}\left(c_n e^{\mu_n x} + d_n e^{-\mu_n x}\right).$$

(*iii*) Sketch the initial temperature profile. [3]

(*iv*) Neglecting unphysical solutions now let the general solution take the form [5]

$$f(x,t) = C + Dx + \sum_{n=1}^{\infty} e^{-\kappa\lambda_n^2 t}\left(a_n \cos\lambda_n x + b_n \sin\lambda_n x\right).$$

By first applying the boundary condition at $x = 0$ and then at $x = 1$, show that the general solution has the form

$$f(x,t) = \sum_{n=1}^{\infty} b_n e^{-\kappa n^2 \pi^2 t} \sin(n\pi x).$$

(*v*) By applying the initial condition show that the solution is given by [5]

$$f(x,t) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2} e^{-\kappa n^2\pi^2 t} \sin(n\pi x).$$

4. Let the matrices $\underline{R_1}(\theta)$ and $\underline{R_2}(\varphi)$ be given by

$$\underline{R_1} = \begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \quad \text{and} \quad \underline{R_2} = \begin{pmatrix} -\cos\varphi & -\sin\varphi \\ \sin\varphi & -\cos\varphi \end{pmatrix}.$$

(*i*) What are the determinants of the two matrices? [5]

(*ii*) Using $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$ and $\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \sin\beta\cos\alpha$, express and simplify $\underline{R_1}\,\underline{R_2}$. What is its determinant? [6]

(*iii*) Find $\underline{R_1}^{-1}$ and $\underline{R_2}^{-1}$ and hence or otherwise find $\left(\underline{R_1}\,\underline{R_2}\right)^{-1}$. [5]

(*iv*) Are the two matrices, $\underline{R_1}$ and $\underline{R_2}$ symmetric? Show that $\underline{R_1}\,\underline{R_2} - \underline{R_2}\,\underline{R_1} = \mathbf{0}$. [6]

(*v*) Show that $\underline{R_1}(-\theta) = \underline{R_1}^{-1}(\theta)$ and $\underline{R_2}(-\varphi) = \underline{R_2}^{-1}(\varphi)$ [3]