MATH6502 Sample Exam Paper 2007-08
For those students taking the half course (30 lectures, quarter-unit).

All questions may be attempted but only marks obtained on the best **three** *solutions will count.*
The use of an electronic calculator **is** *permitted in this examination.*

1. (a) State what we mean when we say that $f(x)$ is an even function. State also what we mean when we say that $f(x)$ is an odd function.

   (b) For each of the following functions, say whether it is even, odd, both or neither: (i) $\cos(nx)$, (ii) $\sin(nx)$, (iii) e^x, (iv) 1, (v) x.

   (c) Using the fact that $e^{i\theta} = \cos\theta + i\sin\theta$, find $\int_0^{2\pi} e^x \{\cos(nx) + i\sin(nx)\}\,dx$, where n is an integer. Use this to show that

$$\int_0^{2\pi} e^x \cos(nx)\,dx = \frac{e^{2\pi} - 1}{(1+n^2)}$$

   and find

$$\int_0^{2\pi} e^x \sin(nx)\,dx.$$

   (d) Recall that the Fourier series representation for a periodic function $f(x)$ which has period 2π is given by

$$F(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

   where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\cos(nx)\,dx, \\ b_n &= \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)\,dx. \end{aligned}$$

   Sketch a graph of the function f that is periodic of period 2π and is defined by $f(x) = e^x$ for $0 \le x < 2\pi$. Then find its Fourier series.

   (e) What value does your series take at $x = \pi$? Use this fact to compute

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(1+n^2)}.$$

2. (a) Show that the general solution to the heat equation in one dimension:

$$\frac{\partial f}{\partial t} = \kappa \frac{\partial^2 f}{\partial x^2}$$

is

$$f(x,t) = Cx + D + \sum_n \exp\left[-\kappa\lambda_n^2 t\right]\left(a_n \cos\lambda_n x + b_n \sin\lambda_n x\right)$$
$$+ \sum_n \exp\left[\kappa\mu_n^2 t\right]\left(c_n \exp\left[\mu_n x\right] + d_n \exp\left[-\mu_n x\right]\right).$$

(b) Find the temperature for all times $t \geq 0$ in a rod of length L and conductivity κ whose ends are held at temperature zero ($f(0,t) = 0$ and $f(L,t) = 0$) and whose initial temperature varies linearly along its length so that $f(x,0) = x$.

You may use without proof the fact that the Fourier sine series for the function x between 0 and L is

$$x = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2L}{n\pi} \sin\left(\frac{n\pi x}{L}\right), \qquad \text{i.e. } b_n = (-1)^{n+1}\frac{2L}{n\pi}.$$

3. You are given the matrix:

$$\underline{\underline{A}} = \begin{pmatrix} 1 & 5 & 2 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{pmatrix}.$$

(a) Find the determinant of $\underline{\underline{A}}$.

(b) **Without** further calculation, but with an explanation in each case, give the determinants of the following matrices by relating them to the matrix $\underline{\underline{A}}$:

$$\text{(i) } \underline{\underline{B}} = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 2 & 14 \\ 0 & -3 & 4 \end{pmatrix} \qquad \text{(ii) } \underline{\underline{C}} = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 1 & 7 \\ 0 & -3 & 4 \end{pmatrix}$$

$$\text{(iii) } \underline{\underline{D}} = \begin{pmatrix} 1 & 5 & 2 \\ 0 & -3 & 4 \\ 1 & 1 & 7 \end{pmatrix} \qquad \text{(iv) } \underline{\underline{E}} = \begin{pmatrix} 1 & 1 & 0 \\ 5 & 1 & -3 \\ 2 & 7 & 4 \end{pmatrix}.$$

(c) Find the inverse of $\underline{\underline{A}}$. What is its determinant?

(d) Using your answer from part (c), solve the linear system $\underline{\underline{A}}\,\underline{x} = \underline{b}$ where

$$\underline{b} = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}.$$

4. (a) A set of simultaneous linear algebraic equations can be written in the form $\underline{\underline{A}}\,\underline{x} = \underline{b}$, where $\underline{\underline{A}}$ is a square matrix of coefficients. What are the different possibilities for the number of solutions that this system can have? State under what conditions a unique solution exists.

(b) Using Gaussian elimination, find the value of t for which the following system:

$$
\begin{array}{rcrcrcrcl}
w & + & x & - & y & + & 3z & = & 4 \\
-w & + & x & + & 5y & - & 3z & = & 4 \\
 & & x & + & 2y & + & z & = & 5 \\
w & + & x & - & y & + & 2z & = & t
\end{array}
$$

has a solution. Find the most general solution to the problem in that case.

(c) What is pivoting in the context of Gaussian elimination? Why might we want to use it in practice?