Mech1010: Integration
Problem sheet 1

Section A

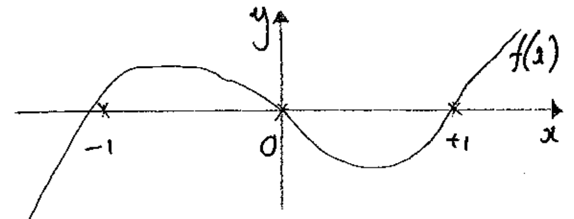
1/ (i) $\int_{-3}^{+3} x^2\,dx = \frac{1}{3}\left[x^3\right]_{-3}^{+3} = \frac{1}{3}(27 - -27) = \frac{54}{3} = 18$

(ii) $\int_{-1}^{+1} x^3 + x\,dx = \left[\frac{1}{4}x^4 + \frac{1}{2}x^2\right]_{-1}^{+1} = \left(\frac{1}{4}+\frac{1}{2}\right) - \left(\frac{1}{4}+\frac{1}{2}\right) = 0$

2/ $f(x) = 2x(x-1)(x+1)$ roots are $x=0$, $x=1$, $x=-1$.

$f'(x) = 2(3x^2-1) \Rightarrow$ stationary points at $x = \pm\frac{1}{\sqrt{3}}$

$f''(x) = 12x$ Thus at $f''(+\frac{1}{\sqrt{3}}) > 0$ minimum, $f''(-\frac{1}{\sqrt{3}}) < 0$ maximum.



$\int_{-1}^{+1} f(x)\,dx = \int_{-1}^{+1} 2x^3 - 2x\,dx$

$= \left[\frac{1}{2}x^4 - x^2\right]_{-1}^{+1} = \left(\frac{1}{2}-1\right) - \left(\frac{1}{2}-1\right)$

$= 0.$

The Area is given by $\left|\int_{-1}^{0} f(x)\cdot dx\right| + \left|\int_{0}^{1} f(x)\,dx\right|$

$= \left|\int_{-1}^{0} 2x^3 - 2x\,dx\right| + \left|\int_{0}^{1} 2x^3 - 2x\,dx\right|$

$= \left|\left[\frac{1}{2}x^4 - x^2\right]_{-1}^{0}\right| + \left|\left[\frac{1}{2}x^4 - x^2\right]_{0}^{1}\right| = 1.$

3/(i) $I = \int \sin(5x+3)\,dx$, let $5x+3 = u \Rightarrow \frac{du}{dx} = 5 \Rightarrow dx = \frac{du}{5}$, so $\frac{1}{5}\int \sin u\,du = I$.

$= -\frac{1}{5}\cos u + c = -\frac{1}{5}\cos(5x+3) + c.$

(ii) $\int (x^6 + 6x^5 + 15x^4 + 20x^3 + 15x^2 + 6x + 1)\,dx = \int (1+x)^6\,dx = \frac{1}{7}(1+x)^7 + c$

(iii) $\int (x+1)^7\,dx = \frac{1}{8}(1+x)^8 + c$

(iv) $\int \frac{dx}{\sqrt{16+4x^2}} = \frac{1}{2}\int \frac{dx}{\sqrt{2^2+x^2}} = \frac{1}{2}\sinh^{-1}\left(\frac{x}{2}\right) + c$

(iv) $\int \frac{dx}{\sqrt{x^2+2x+17}} = \int \frac{dx}{\sqrt{(x+1)^2+4^2}}$ let $x+1 = u$, $dx = du$, so

$= \int \frac{du}{\sqrt{u^2+4^2}} = \sinh^{-1}\left(\frac{u}{4}\right) + c = \sinh^{-1}\left(\frac{x+1}{4}\right) + c$

Section B

4/ $\int \left(2x^2 - \frac{1}{x}\right)^2 dx = \int \left(4x^2 - 4x + \frac{1}{x^2}\right) dx$

$= \frac{4x^3}{3} - 2x^2 - \frac{1}{x} + C$

5/ $\int_{-a}^{b} f(x)\, dx = \int_{-a}^{b} x(x+a)(x-b)\, dx = \int_{-a}^{b} x^3 + x^2(a-b) - xab . dx$

$= \left[\frac{1}{4}x^4 + \frac{1}{3}x^3(a-b) - \frac{1}{2}x^2 ab\right]_{-a}^{b}$

$= \left(\frac{1}{4}b^4 + \frac{1}{3}b^3(a-b) - \frac{1}{2}b^3 a\right) - \left(\frac{1}{4}a^4 - \frac{a^3}{3}(a-b) - \frac{a^3 b}{2}\right)$

$= \frac{1}{4}(b^4 - a^4) + \frac{1}{3}(b^3 + a^3)(a-b) + \frac{ab}{2}(a^2 - b^2)$

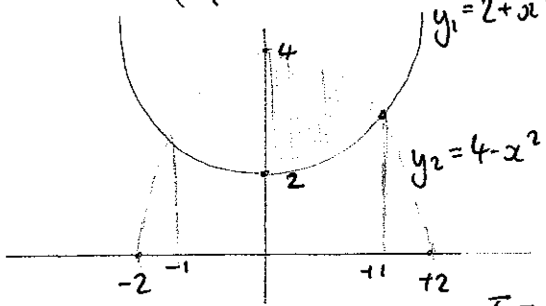
$= \frac{1}{4}(b^2 - a^2)(b^2 + a^2) + \frac{1}{3}(b^3 + a^3)(a-b) + \frac{ab}{2}(a+b)(a-b)$

$= \frac{1}{4}(b-a)(b+a)(b^2+a^2) + \frac{1}{3}(b^3+a^3)(a-b) + \frac{ab}{2}(a+b)(a-b)$

$= \left(-\frac{1}{4}(b+a)(b^2+a^2) + \frac{1}{3}(b^3+a^3) + \frac{ab}{2}(a+b)\right)(a-b) = \left(\frac{b^3}{6} + \frac{a^3}{6} + \frac{a^2 b}{4} + \frac{b^2 a}{4}\right)(a-b)$

So integral is positive if $a > b$
negative if $b > a$.

6/



$y_1 = y_2 \Rightarrow 2 + x^2 = 4 - x^2$

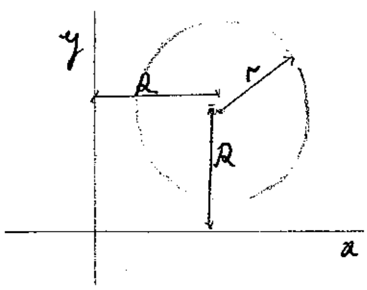
$2x^2 = 2 \Rightarrow x = \pm 1$

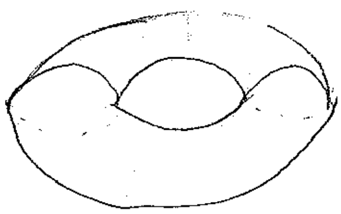
By symmetry let the area be

$I = 2\int_0^1 (4 - x^2) - (2 + x^2)\, dx = 2\int_0^1 2(1 - x^2)\, dx$

$= 4\left[x - \frac{1}{3}x^3\right]_0^1 = 4\left(1 - \frac{1}{3}\right) = 8/3.$

7/





circle defined by $(x-R)^2 + (y-R)^2 = r^2$

Area of Circle: πr^2

Volume is area rotated about

$2\pi R$, i.e

$V = 2\pi^2 r^2 R$