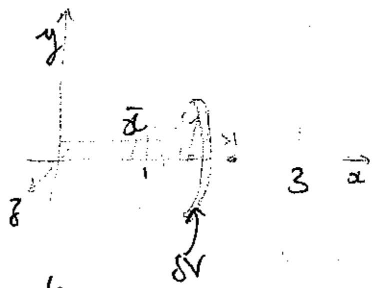
MECH 1010: INTEGRATION
Problem Sheet 2

Section A

1/ By symmetry the $\bar{y}=0, \bar{z}=0$



A suitable volume element is $\delta V = \pi y^2 \delta x$, where $y = 2x-1$, ie. $\delta V = \pi(2x-1)^2 \delta x$.

Thus the first moment about the x-axis, given by

$\delta M_V^x = x\,\delta V = \pi x(2x-1)^2 \delta x$, thus $M_V^x = \pi \int_1^3 x(2x-1)^2 dx$

$= \pi \int_1^3 4x^3 - 4x^2 + x \, dx$

$= \pi\left[x^4 - \frac{4x^3}{3} + \frac{x^2}{2}\right]_1^3 =$

$= \pi\left(81 - \frac{4.27}{3} + \frac{9}{2}\right) - \pi\left(1 - \frac{4}{3}.1 + 1/2\right)$

$= \pi(81 - 36 + 9/2) - \pi(3/2 - 4/3)$

$= \pi(45 + 9/2) - \pi/6$

$= \pi(99/2 - 1/6)$

$= \pi(297-1)/6 = \pi\, 296/6$

$= \pi 148/3$

The volume is $\pi \int_1^3 y^2 . dx = \pi \int_1^3 (2x-1)^2 dx$

$= \pi \int_1^3 4x^2 - 4x + 1 \, dx$

$= \pi\left[\frac{4x^3}{3} - 2x^2 + x\right]_1^3$

$= \pi\left(4.\frac{27}{3} - 2.9 + 3\right) - \pi\left(\frac{4}{3}.1 - 2 + 1\right)$

$= \pi\left((36 - 18 + 3) - \left(\frac{4}{3} - 2 + 1\right)\right)$

$= \pi(21 - 1/3) = \pi(62/3)$

Thus $\bar{x} = M_V^x / V = \dfrac{\pi 148/3}{\pi 62/3} = 148/62.$

2/(i) $\delta A = 2x\,\delta y$

$\delta I_{AO} = y^2 \delta A = 2x y^2 \delta y.$

$\Rightarrow I_{AO} = 2\int_{-R}^{+R} y^2 x \, dy = 4\int_0^R y^2 x \, dy.$

(ii) Given $x = R\cos\theta$, $y = R\sin\theta$, $\frac{dy}{d\theta} = R\cos\theta$, $\frac{dx}{d\theta} = -R\sin\theta.$

$dy = R\cos\theta . d\theta$

Also when $y=0$, then $\theta = \pi/2$ and when $y = R$, $\theta = 0$.

$$I_{AO} = 4\int_0^{\pi/2} (R\sin\theta)^2 R\cos\theta \,.\, R\cos\theta \, d\theta$$

$$= 4R^4 \int_0^{\pi/2} \sin^2\theta \,.\, \cos^2\theta = \frac{4R^4}{4}\int_0^{\pi/2} (1-\cos 2\theta)(1+\cos 2\theta)\, d\theta$$

$$= \frac{R^4}{2}\int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= \frac{R^4}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}$$

$$= \frac{\pi R^4}{4}$$

ii) Applying the parallel axis theorem

$$I_h = I_{OA} + Ah^2 = \frac{\pi R^4}{4} + \pi R^2 h^2.$$

Section B

3. Either by the Cartesian form $(x-R)^2 + (y-R)^2 = r^2$,
Polar $r = r,\ \theta = \theta$ where $r = $ const.
or Parametric $x = r\sin t,\ y = r\cos t.$

The arc-length element can be defined, integrated and rotated around the x- or y-axis.

$$A = (2\pi r)(2\pi R) = 4\pi^2 r R$$

4. By symmetry the length from -1 to 0 is equal to 0 to 1.
The length is given by $s = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$

$y = 6.22 + \cosh x$, $\frac{dy}{dx} = \cancel{2\cosh x}\ \sinh x$

$$S = 2\int_0^1 \sqrt{(1 + \cancel{4\cosh^2 x}\ \sinh^2 x)}\, dx = 2\int_0^1 \sqrt{1+\sinh^2 x}\, dx$$

$$= 2\int_0^1 \cosh x \, dx$$

$$= 2\left[\sinh x\right]_0^1 = 2\sinh 1.$$