Mech 2010: Integration
Sheet 3

Section A

1/ (i) From Notes $\int_0^1 x^3 e^x dx = I_3 = [x^3e^x]_0^1 - 3I_2$

$= [x^3e^x]_0^1 - 3\int_0^1 x^2 e^x dx$

$I_2 = [x^2e^x]_0^1 - 2I_1$, $I_1 = [xe^x]_0^1 - 1.I_0$, $I_0 = \int_0^1 x^0 e^x . dx = \int_0^1 e^x dx = e^1 - e^0 = e^1 - 1$

$\Rightarrow I_1 = e^1 - (e^1 - 1) = 1$, $I_2 = e^1 - 2$, $I_3 = e^1 - 3(e^1 - 2) = \cancel{6 - 3e^1}$ $6 - 2e^1$

(ii) From Notes $I_4 = \int_0^{\pi/2} \cos^4\theta . d\theta$

$= \left[\frac{1}{4}\cos^3\theta \sin\theta + \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta\right]_0^{\pi/2}$

$= \left(\frac{1}{4}\underbrace{\cos^3\frac{\pi}{2}}_{0} \sin\frac{\pi}{2} + \frac{3}{8}\underbrace{\cos\frac{\pi}{2}}_{0}\sin\frac{\pi}{2} + \frac{3}{8}\pi/2\right) - \frac{1}{4}\cos 0 \underbrace{\sin 0}_{0} - \frac{3}{8}\cos 0 \underbrace{\sin 0}_{0} \underbrace{\tfrac{3}{8}0}_{0}$

$= \frac{3\pi}{16}$.

2/ By Partial Fractions $\frac{x}{1+x} = \frac{A(1+x)}{1+x} + \frac{B}{1+x}$ or note that $\frac{x}{1+x} = \frac{1+x}{1+x} - \frac{1}{1+x}$

so that $\int \frac{x}{1+x} dx = \int 1 . dx - \int \frac{1}{1+x} dx = x - \ln|1+x| + c$.

3/ (i) $f(x) = xe^x$, $f'(x) = e^x + xe^x$ by chain rule.

$\int f(x) f'(x) . dx$ is in standard form and we know that $\int f(x)f'(x) dx = \frac{1}{2}f^2(x) + c$, i.e explicitly $\int e^x(1+x) . xe^x dx$

$= \int x(1+x)e^{2x} dx = \frac{1}{2}x^2 e^{2x} + c$.

(ii) $\int \frac{f(x)}{f'(x)} . dx$ is <u>not</u> in standard form, but is given by

$\int \frac{xe^x}{(1+x)e^x} = \int \frac{x}{1+x} dx = x - \ln|1+x| + c$ from (2).

Section B

4/ $\int_{-1}^{+1} \frac{\ln(3+2x)}{3+2x}$ let $3+2x = u$, then when $\begin{cases} x=-1, & u=1 \\ x=1, & u=5. \end{cases}$

$\Rightarrow \int_1^5 \frac{\ln u}{u} . du$. now let $\ln u = v$, so that $\frac{dv}{du} = \frac{1}{u}$ i.e. $dv = \frac{du}{u}$.

$u = e^v$

$\Rightarrow \int_0^{\ln 5} v . dv = \left[\frac{1}{2}v^2\right]_0^{\ln 5} = \frac{1}{2}(\ln 5)^2$

(i) By partial fractions,

$$\frac{3x^2-1}{x(x+1)(x-1)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-1} \Rightarrow 3x^2-1 = A(x^2-1) + Bx(x-1) + Cx(x-1)$$

$\Rightarrow x^2(A+B+C) + x(C-B) - A = 3x^2-1$. Equating powers of x gives

$$\left.\begin{array}{l} A=+1 \\ B=C \\ A+B+C=3 \Rightarrow B=1, C=1, A=1. \end{array}\right\} \int \frac{3x^2-1}{x(x+1)(x-1)}dx = \int \frac{1}{x}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{1}{x-1}\,dx$$

$$= \ln|x| + \ln|x+1| + \ln|x-1| + c.$$

(ii) On the substitution $u = \tan x$

$$I = \int \frac{dx}{3\sin^2 x - 5\cos^2 x} = \int \frac{(1/1+u^2)\,du}{(3u^2-5)/(1+u^2)} = \int \frac{du}{3u^2-5}$$

$$= \frac{1}{3}\int \frac{du}{u^2-(5/3)} = \frac{1}{3}\int \frac{du}{u^2-(\sqrt{5/3})^2} = \frac{1}{3}\int \frac{du}{(u-\sqrt{5/3})(u+\sqrt{5/3})}$$ (solve via partial fractions or use standard form in notes.)

$$= \frac{1}{2.3.\sqrt{5/3}} \ln\left|\frac{u-\sqrt{5/3}}{u+\sqrt{5/3}}\right| + c = \frac{1}{2\sqrt{15}} \ln\left|\frac{\tan x - \sqrt{5/3}}{\tan x + \sqrt{5/3}}\right| + c.$$

$I_{n,m} = \int \sin^n x \cos^m x\,dx$. Let $\left\{ \begin{array}{l} u = \sin^{n-1}x\cos^m x, \quad dv = \sin x \\ du = (n-1)\sin^{n-2}x\cos^{m+1}x - m\sin^n x\cos^{m-1}x \\ v = -\cos x \end{array}\right.$

ing $\int u\,dv = uv - \int v\,du$,

is: $I_{n,m} = -\sin^{n-1}x\cos^{m+1}x - (n-1)\int \sin^{n-2}x\cos^m x(\sin^2 x - 1)\,dx - mI_{n,m}$

$$= -\sin^{n-1}x\cos^{m+1}x - (n-1)I_{n,m} - mI_{n,m} + (n-1)\int \sin^{n-2}x\cos^m x\,dx$$

$$\Rightarrow I_{n,m}(1+m+(n-1)) = -\sin^{n-1}x\cos^{m+1}x + (n-1)\int \sin^{n-2}x\cos^m x\,dx$$

$$I_{n,m} = -\frac{\sin^{n-1}x\cos^{m+1}x}{n+m} + \frac{n-1}{n+m}\int \sin^{n-2}x\cos^m x\,dx.$$

r the other part split the integral as $\underbrace{\cos x}_{dv}.(\underbrace{\cos^{m-1}x\sin^n x}_{u})$ and apply
tegration by parts.

/ 10 equal parts: let $d = 1/10$.

By Trapezium rule: $\frac{1}{2}\cdot\frac{1}{10}(1 + 2\cdot(0.990050 + 0.960789 + 0.913913 + 0.852144$
$+ 0.778801 + 0.697676 + 0.612626 + 0.527292$
$+ 0.444858) + 0.367879) = \underline{0.746211}$

By Simpson's rule: $\frac{1}{3}\cdot\frac{1}{10}(1 + 4.(0.990050 + 0.913913 + 0.778801 + 0.612626$
$+ 0.44858) + 2.(0.960789 + 0.852144 +$
$0.697676 + 0.527292) + 0.367879)$

$= 0.746825$