MECH1010 : Modelling and Analysis in Engineering I
Integration

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Contents

*This document can be downloaded from: http://www.ucl.ac.uk/~ucesdsi/teaching.html

Recommended Reading

- K. A. Stroud, *Engineering Mathematics* London: Palgrave Macmillan, 6th Revised edition.

1 Introduction

The preliminary stages of this course are intended to reacquaint you with standard material in order to investigate the applications of integration in engineering .

There are two types of integration - definite and indefinite. An indefinite integral is the anti-derivative of a function $f(x)$, i.e. let

$$f'(x) = \frac{\mathrm{d}f(x)}{\mathrm{d}x}$$

then

$$\int f'(x)\ \mathrm{d}x = f(x) + c$$

where c is the constant of integration. The constant of integration appears as information is lost when a constant is differentiated. However, the constant of integration can be determined by knowing the value of the function at a given point such as the initial conditions of the function, i.e. $f(0)$.

$$\begin{tikzcd} f'(x) \arrow[r, bend left, "\text{integration}"] & f(x) \arrow[l, bend left, "\text{differentiation}"] \end{tikzcd}$$

Example 1.1. *If* $f(x) = x^2 + 4x + 5$ *so then* $f(0) = 5$. *On differentiation* $f'(x) = 2x + 4$. *On integration* $\int f'(x)\ \mathrm{d}x = x^2 + 4x + c$. *In order to ascertain the value of* c *we need to set* $x = 0$. *Thus* $c = 5$. □

2 Standard Integrals

There are no infallible rules by which we can ascertain the indefinite integral of any given function. We are best served by the combination of experience and a systematic approach to the problem. Accordingly, the recognition of standard integrals is perhaps the most important skill required in integral calculus. You should, therefore be thoroughly familiar with, or make yourself thoroughly familiar with, the following standard results:

$$\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c \tag{1a}$$

$$\int \frac{1}{x} \, \mathrm{d}x = \ln|x| + c \tag{1b}$$

$$\int a^x \, \mathrm{d}x = \frac{a^x}{\ln|a|} + c \tag{1c}$$

$$\Rightarrow \int e^x \, \mathrm{d}x = e^x + c \tag{1d}$$

$$\int \sin x \, \mathrm{d}x = -\cos x + c \tag{1e}$$

$$\int \cos x \, \mathrm{d}x = \sin x + c \tag{1f}$$

$$\int \sec^2 x \, \mathrm{d}x = \tan x + c \tag{1g}$$

$$\int \mathrm{cosec}^2 x \, \mathrm{d}x = -\cot x + c \tag{1h}$$

$$\int \sinh x \, \mathrm{d}x = \cosh x + c \tag{1i}$$

$$\int \cosh x \, \mathrm{d}x = \sinh x + c \tag{1j}$$

$$\int \mathrm{sech}^2 x \, \mathrm{d}x = \tanh x + c \tag{1k}$$

$$\int \mathrm{cosech}^2 x \, \mathrm{d}x = -\coth x + c \tag{1l}$$

These standard integrals are directly obtained by writing the standard derivatives in reverse. Another, family of standard integrals are the logarithmic and inverse trigonometric forms

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \sin^{-1} \frac{x}{a} + c \tag{2a}$$

$$\int \frac{1}{\sqrt{a^2 + x^2}} \, \mathrm{d}x = \sinh^{-1} \frac{x}{a} + c \tag{2b}$$

$$-\int \frac{1}{\sqrt{a^2 - x^2}} \, \mathrm{d}x = \cos^{-1} \frac{x}{a} + c \tag{2c}$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, \mathrm{d}x = \cosh^{-1} \frac{x}{a} + c \tag{2d}$$

$$\int \frac{1}{a^2 + x^2} \, \mathrm{d}x = \frac{1}{a} \tan^{-1} \frac{x}{a} + c \tag{2e}$$

$$\int \frac{1}{a^2 - x^2} \, \mathrm{d}x = \frac{1}{a} \tanh^{-1} \frac{x}{a} + c \tag{2f}$$

$$\int \frac{1}{x^2 - a^2} \, \mathrm{d}x = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + c \tag{2g}$$

2.1 Extending the Standard Integral Forms

A polynomial expression can be integrated term by term, including each constant of integration into a single term

Example 2.1. *Consider the indefinite integral*

$$\int \left(8x^3 + 36x^2 + 54x + 27\right)\,\mathrm{d}x = \int 8x^3\,\mathrm{d}x + \int 36x^2\,\mathrm{d}x + \int 54x\,\mathrm{d}x + \int 27\,\mathrm{d}x$$

$$\int 8x^3\,\mathrm{d}x = 2x^4 + c_1$$

$$\int 36x^2\,\mathrm{d}x = 12x^3 + c_2$$

$$\int 54x\,\mathrm{d}x = 27x^2 + c_3$$

$$\int 27\,\mathrm{d}x = 27x + c_4$$

$$= 2x^4 + 12x^3 + 27x^2 + 27x + c$$

where $c = c_1 + c_2 + c_3 + c_4$. □

The standard integral form can be readily extended by a linear change of variables

Example 2.2. *Noting that*

$$f(x) = \left(8x^3 + 36x^2 + 54x + 27\right) = (2x+3)^3$$

Then implies that the integral

$$\int f(x)\,\mathrm{d}x = \int \left(8x^3 + 36x^2 + 54x + 27\right)\,\mathrm{d}x = \int (2x+3)^3\,\mathrm{d}x$$

Now let $h = 2x + 3$. *Hence,* $\frac{\mathrm{d}h}{\mathrm{d}x} = 2$. *Thus* $\mathrm{d}x = \frac{1}{2}\mathrm{d}h$, *so the integral to solve now becomes*

$$\begin{aligned}\int (2x+3)^3\,\mathrm{d}x &= \int h^3\,\mathrm{d}x\\ &= \frac{1}{2}\int h^3\,\mathrm{d}h\\ &= \frac{1}{6}h^4 + c\\ &= \frac{1}{6}(2x+3)^4 + c.\end{aligned}$$

□

Example 2.3. *Find the indefinite integral*

$$\int \frac{1}{4x+3}\,\mathrm{d}x$$

thus let $h = 4x + 3$, *then* $\frac{\mathrm{d}h}{\mathrm{d}x} = 4$, *so that* $\mathrm{d}x = \frac{1}{4}\mathrm{d}h$

$$\begin{aligned}\int \frac{1}{4x+3}\,\mathrm{d}x &= \int \frac{1}{h}\,\mathrm{d}x\\ &= \frac{1}{4}\int \frac{1}{h}\mathrm{d}h\\ &= \frac{1}{4}\ln|4x+3| + c.\end{aligned}$$

□

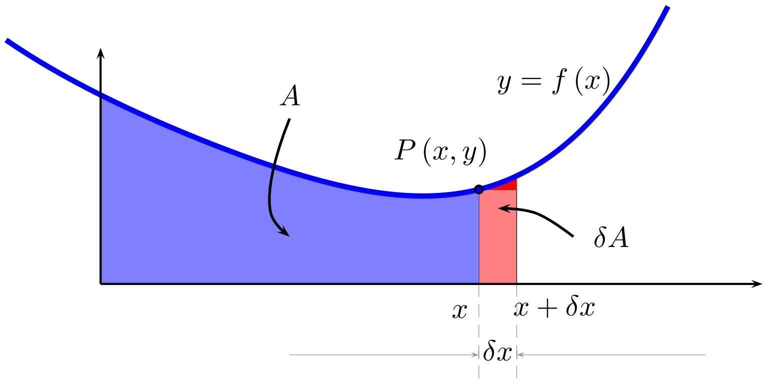


Figure 1: Integration: as δx decreases so does the error

Example 2.4. *Find the indefinite integral*

$$\int \sin(2x+1)\,\mathrm{d}x$$

thus let $h = 2x + 1$, *then* $\frac{\mathrm{d}h}{\mathrm{d}x} = 2$, *hence* $\mathrm{d}x = \frac{1}{2}\mathrm{d}h$

$$\int \sin(2x+1)\,\mathrm{d}x = -\frac{1}{2}\cos(2x+1) + c. \qquad \square$$

3 Definite Integrals & Applications

3.1 Areas Under Curves

The area bounded by a curve and the x-axis may be found from reducing the problem to an elemental form. Consider a point P on the curve $y = f(x)$ at x, that is $P(x, y = f(x))$. We can define a rectangular elemental of area to be given by

$$\delta A \approx f(x)\,\delta x.$$

This implies that

$$f(x) \approx \frac{\delta A}{\delta x}.$$

If the width of the rectangle tends to zero, that is $\delta x \to 0$, then δA becomes equal to the area bounded by the curve and the x-axis between x and δx. Thus

$$f(x) = \frac{\mathrm{d}A}{\mathrm{d}x}$$

Thus, $f(x)$ is equal to the derivative of the area A enclosed between the curve and the x-axis with respect to x. Consequently the area A corresponds to the integral of the function $f(x)$.

A definite integral is the area under the curve given by $f(x)$ between two values a and b, say. The integral is called a definite integral as it is a integral with defined limits. The constant of integration disappears in the subtraction leaving a numerical value. Let $F(x)$ denote the integral of $f(x)$, that is

$$F(x) = \int f(x)\,\mathrm{d}x$$

then

$$\int_a^b f(x)\, \mathrm{d}x = F(b) - F(a).$$

Hence

$$\int_a^a f(x)\, \mathrm{d}x = F(a) - F(a) = 0.$$

We can also infer that

$$-\int_a^b f(x)\, \mathrm{d}x = F(a) - F(b) = \int_b^a f(x)\, \mathrm{d}x.$$

Thus if we reverse the integration from b to a we are simply changing the sign of the interval over which the integration is performed. If we are integrating from a to c, we can sub-divide the problem at an intermediate point b

$$\int_a^c f(x)\, \mathrm{d}x = \int_a^b f(x)\, \mathrm{d}x + \int_b^c f(x)\, \mathrm{d}x = F(b) - F(a) + F(c) - F(b) = F(c) - F(a).$$

Care is required though, for example

$$\begin{aligned}\int_0^1 4x(x-1)(x-2)\, \mathrm{d}x &= \int_0^1 4x^3 - 12x^2 + 8x\, \mathrm{d}x \\ &= \left[x^4 - 4x^3 + 4x^2\right]_0^1 \\ &= 1 - 0 \\ &= 1.\end{aligned}$$

We may now solve the same integral over a larger range

$$\begin{aligned}\int_0^2 4x(x-1)(x-2)\, \mathrm{d}x &= \int_0^2 4x^3 - 12x^2 + 8x\, \mathrm{d}x \\ &= \left[x^4 - 4x^3 + 4x^2\right]_0^2 \\ &= 0 - 0 \\ &= 0.\end{aligned}$$

Can the area be zero? How is this possible? If the curve has a negative value, the integral gives a negative area. Therefore, care must be taken when part of the area is above the x-axis and part below it, in order to avoid areas of opposite sign cancelling each other out. Thus, it is wise to sketch the curve before carrying out the integration and, if necessary, to calculate the positive and negative parts separately. For the previous case

$$\begin{aligned}A &= \left|\int_0^1 f(x)\, \mathrm{d}x\right| = \left|\int_0^1 4x^3 - 12x^2 + 8x\, \mathrm{d}x\right| = 1, \\ B &= \left|\int_1^2 f(x)\, \mathrm{d}x\right| = \left|\int_1^2 4x^3 - 12x^2 + 8x\, \mathrm{d}x\right| = |-1| = 1\end{aligned}$$

Thus the total area is $A + B = 2$.

3.2 Horizontal Elements

Instead of the area between a curve and the x-axis, we may require the area between a curve and the y-axis. In this case it can be more convenient to choose a thin horizontal element area δA approximated by a rectangle with width δy and length x so that $\delta A \approx x\delta y = f(y)\,\delta y$. The total area required is found from the summation of the horizontal area elements between the given limits. Therefore in the limiting case as $\lim \delta y \to 0$, we have $A = \int_a^b \mathrm{d}A = \int_a^b f(y)\, \mathrm{d}y$.

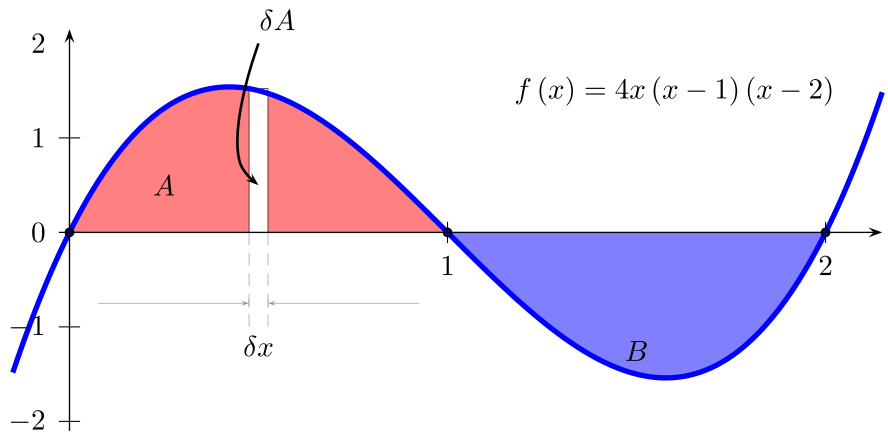


Figure 2: Areas between curves

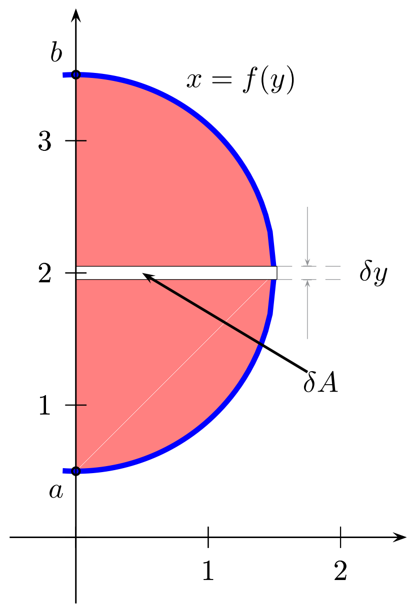


Figure 3: Horizontal elements

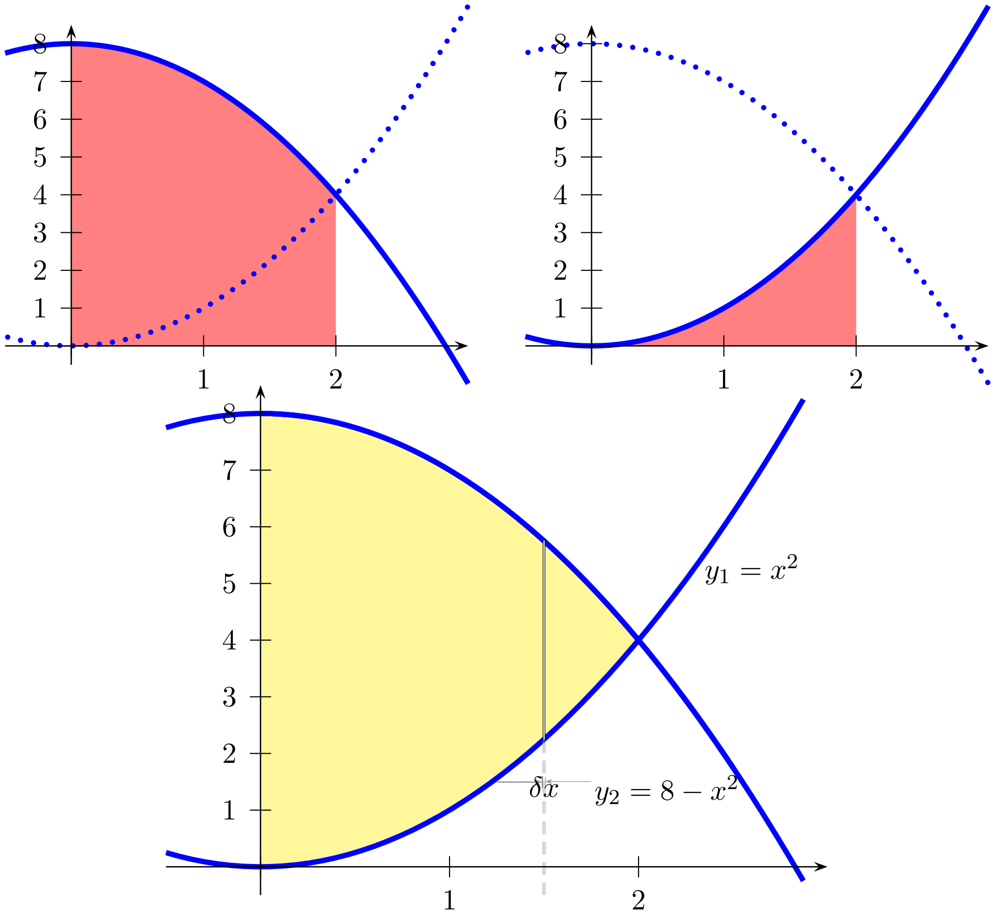


Figure 4: Areas between curves

3.3 Compound Areas

The basic method can readily be extended to more complicated areas and shapes provided that a suitable element can be defined.

Example 3.1. *In order to find the size of the shaded area between $y_1 = x^2$, $y_2 = 8 - x^2$ and the x- and y-axes, we could find the area beneath each of the two curves separately and subtract them or we could define an elemental area δA which approximates an elemental area of width δx and height $y_2 - y_1$. Thus, in the limit* $\lim \delta x \to 0$,

$$\begin{aligned} A &= \int_0^2 y_2 - y_1 \, \mathrm{d}x \\ &= \int_0^2 \left(8 - x^2\right) - x^2 \, \mathrm{d}x \\ &= \int_0^2 8 - 2x^2 \, \mathrm{d}x \\ &= \left[8x - \frac{2}{3}x^3\right]_0^2 \\ &= \frac{32}{3} \end{aligned}$$

□

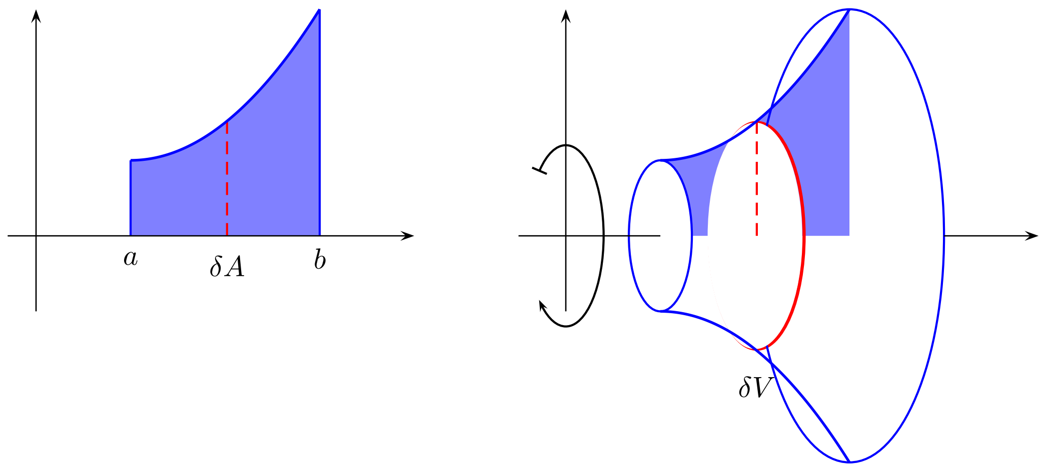


Figure 5: Solids of revolution

4 Solids of Revolution

If an object is rotated about a straight line it forms a three dimensional object known as a solid of revolution. The volume of this three-dimensional object can be found using integration and is known as a volume of integration. The method for calculating the volume of revolution is

(i) Define a rectangular area element with width δx and height $y = f(x)$, i.e $\delta A \approx y\delta x$.

(ii) Rotate it about the x-axis to generate a cylindrical volume element δV

$$\delta V \approx \pi y^2 \delta x = \pi f^2(x)\, \delta x.$$

(iii) The total volume of the solid of revolution V is given by the sum of all the individual elements δV in the limit $\delta x \to 0$

$$V = \int_a^b \mathrm{d}V = \pi \int_a^b y^2\, \mathrm{d}x = \pi \int_a^b f^2(x)\, \mathrm{d}x.$$

Example 4.1. *Consider the volume of revolution formed by rotating the area bounded by the x-axis, the ordinates $x = 1$ and $x = 5$ and the curve $y = 2x + 6$ about the x-axis. Therefore $\delta V \approx \pi y^2 \delta x = \pi (2x+6)^2 \delta x$ The total volume of the solid, V, is the summation of the volume elements δV between $x = 1$ and $x = 5$ as $\delta x \to 0$, that is*

$$\begin{aligned} V &= \int_1^5 \mathrm{d}V \\ &= \pi \int_1^5 (2x+6)^2\, \mathrm{d}x \\ &= \frac{\pi}{6}\left[(2x+6)^3\right]_1^5 = \frac{1792\pi}{3}. \end{aligned}$$ □

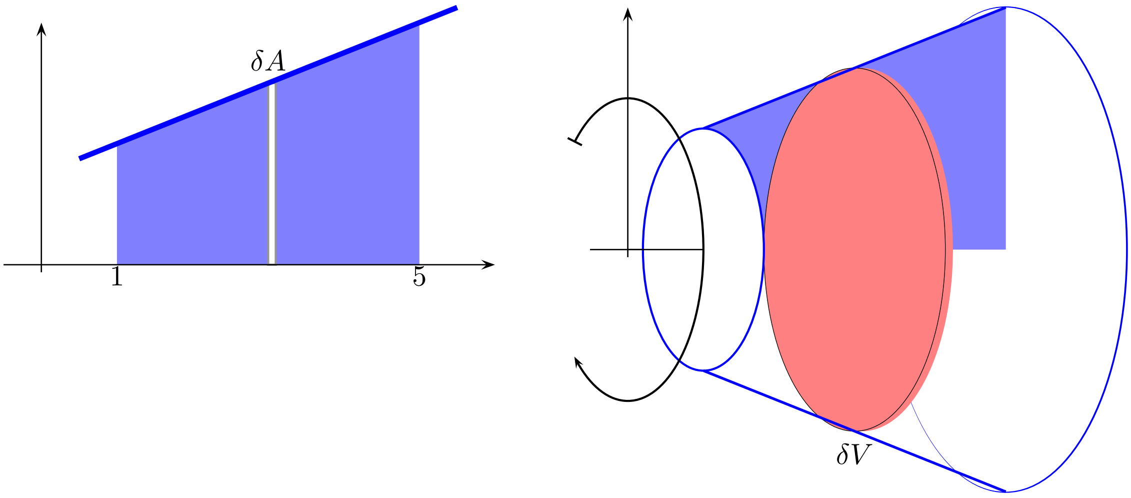


Figure 6: Volume of revolution

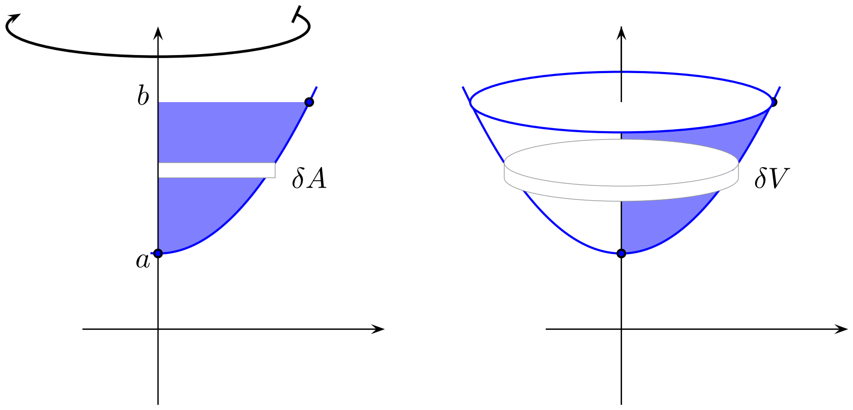


Figure 7: Rotations About the $\boldsymbol{y}$-Axis

4.1 Rotations About the $\boldsymbol{y}$-Axis

If we wish to calculate the volume of revolution by rotating an area between a curve and the y-axis about the y-axis, we can use horizontal rectangular elements with width δy and length $x = f(y)$. The elemental volume is given by

$$\delta V \approx \pi x^2 \delta y = \pi f^2(y)\,\delta y.$$

Again, the total volume is the sum of the elemental volume elements contained within the solid of revolution as volume of each element tends to zero.

$$V = \int_a^b \mathrm{d}V = \pi \int_a^b f^2(y)\,\mathrm{d}y.$$

4.2 Compound Volumes

The basic method of summation is readily extended to more complex solids of revolution. Consider the area $A'ABB'$ rotated about the y-axis to form a solid of revolution. Define a rectangular

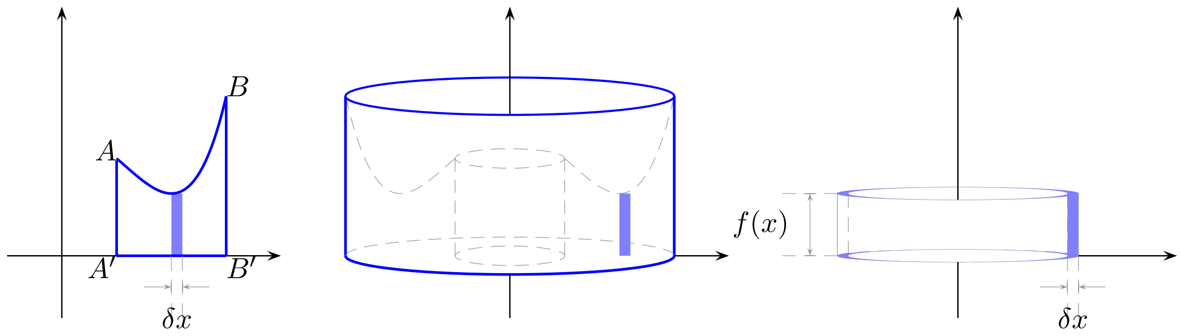


Figure 8: Compound volumes

element of thickness δx and area $\delta A \approx y\delta x$. We can obtain volume elements δV by rotating the area δA around the circumference $2\pi x$. The elemental volume is then given by

$$\delta V \approx 2\pi x y \delta x = 2\pi x f(x)\, \delta x.$$

Once again,

$$V = 2\pi \int_a^b x f(x)\, \mathrm{d}x.$$

Example 4.2. *Find the volume generated when the plane figure bounded by the curve $y = f(x) = x^2 + 5$, the y-axis and the ordinates $x = 1$ and $x = 3$ is rotated about the y-axis through a complete revolution. Thus let a volume element be given by*

$$\delta V \approx 2\pi x \left(x^2 + 5\right) \delta x = 2\pi \left(x^3 + 5x\right) \delta x.$$

Then

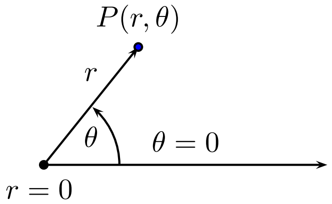
$$\begin{aligned} V &= \int_1^3 \mathrm{d}V \\ &= 2\pi \int_1^3 x^3 + 5x \,\mathrm{d}x \\ &= 2\pi \left[\frac{x^4}{4} + \frac{5x^2}{2}\right]_1^3 \\ &= 80\pi. \end{aligned}$$

□

5 Polar Coordinates

The position of a point, P, in a plane can be expressed in terms of polar coordinates (r, θ), rather than Cartesian coordinates (x, y), i.e. $P = P(r, \theta)$. It is often advantageous to solve problems in this setting.

$$x = r\cos\theta, \qquad r = \sqrt{x^2 + y^2},$$
$$y = r\sin\theta, \qquad \theta = \tan^{-1}\frac{y}{x}.$$



The general approach is identical to that applied in the case of Cartesian coordinates, i.e.

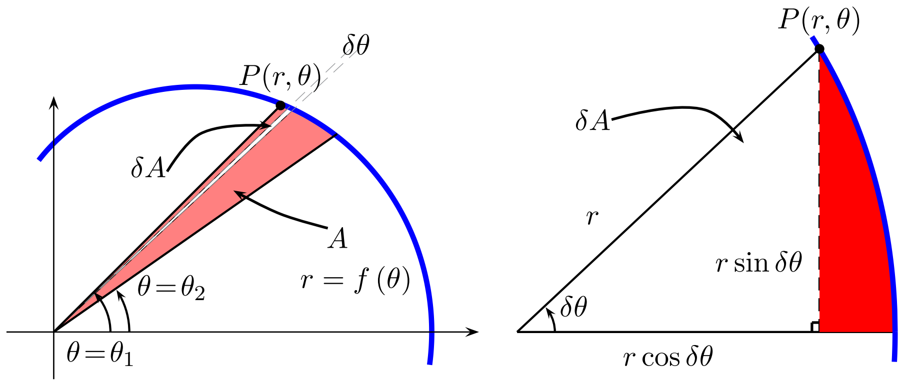


Figure 9: Areas in polar coordinates

(i) Define a small area or volume element.

(ii) Determine the elemental value of the property of interest.

(iii) Sum the elemental values over the total area or volume required.

Consider the area under a curve $r = f(\theta)$ between the limits $\theta = \theta_1$ and $\theta = \theta_2$. In this case we take a thin triangular element δA where

$$\delta A \approx \frac{1}{2} r^2 \sin\delta\theta \cos\delta\theta.$$

As $\delta\theta$ is small we may approximate $\sin\delta\theta \approx \delta\theta$ and $\cos\delta\theta \approx 1$, hence

$$\delta A \approx \frac{1}{2} r^2 \delta\theta.$$

Therefore in the limit as $\delta\theta \to 0$

$$A = \int_{\theta_1}^{\theta_2} \mathrm{d}A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 \mathrm{d}\theta.$$

6 Parametric Equations

In addition to Cartesian and polar coordinates, we are often required to deal with parametric equations. Consider a curve given by the parametric equations

$$x = a\cos\theta \quad \text{and} \quad y = b\sin\theta.$$

This is the parametric form of an ellipse. Often we are required to find the area under the curve between $x = 0$ and $x = a$.

(i) Choose a thin vertical element of height y and width δx. Thus the area is given by $A = \int_0^a y\mathrm{d}x$.

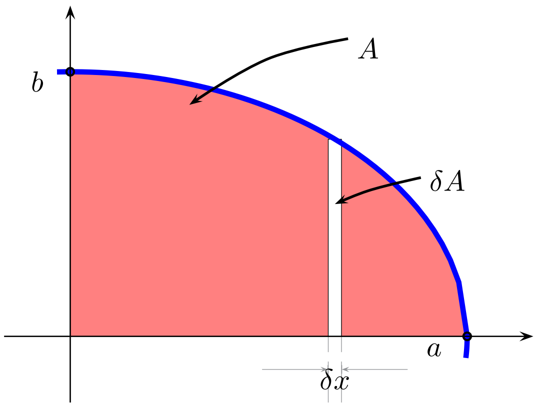
(ii) Express the integral, including the limits of integration, in terms of the parameter θ.

The limits of integration in terms of the parameter θ can easily be found: when $x = 0$ then $\cos\theta = 0$ thus $\theta = \pi/2$. Similarly when $x = a$ then $\theta = 0$. As $x = a\cos\theta$, then $\dfrac{\mathrm{d}x}{\mathrm{d}\theta} = -a\sin\theta$, thus we may say that $\mathrm{d}x = -a\sin\theta\,\mathrm{d}\theta$. Hence the integral becomes

$$A = -ab\int_{\pi/2}^{0} \sin^2\theta\,\mathrm{d}\theta$$

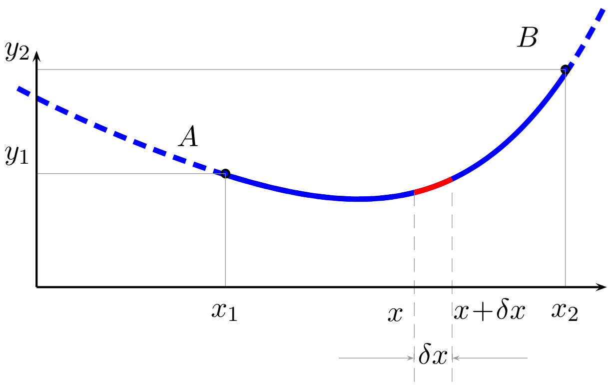
Using the double-angle formula $\sin^2\theta = \dfrac{1}{2}(1-\cos 2\theta)$

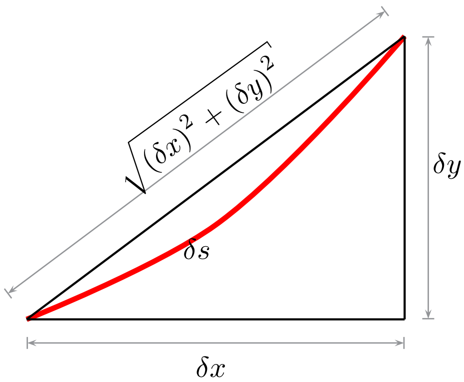
$$\begin{aligned} A &= -\frac{ab}{2}\int_{\pi/2}^{0}(1-\cos 2\theta)\,\mathrm{d}\theta \\ &= -\frac{ab}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_{\pi/2}^{0} \\ &= -\frac{ab}{2}\left(0 - \frac{\pi}{2}\right) \\ &= \frac{\pi ab}{4}. \end{aligned}$$



7 Arc-Lengths & Curved Surface Areas

Consider the general arc AB





Let a small element of the arc be δs. The approximate length of this element if given by $\delta s \approx \sqrt{(\delta x)^2 + (\delta y)^2}$. Hence δs maybe written as

$$\delta s \approx \delta x\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} \quad \text{or} \quad \delta s \approx \delta y\sqrt{1+\left(\frac{\delta x}{\delta y}\right)^2}.$$

Therefore the complete arc-length from A to B is

$$s = \int_{x_1}^{x_2} \sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x \quad \text{or} \quad s = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{\mathrm{d}x}{\mathrm{d}y}\right)^2}\,\mathrm{d}y.$$

If the curve is given by a parametric equation so that $x = x(\theta)$ and $y = y(\theta)$ then the approximate arc-length element is given by

$$\delta s \approx \delta\theta\sqrt{\left(\frac{\delta y}{\delta\theta}\right)^2 + \left(\frac{\delta y}{\delta\theta}\right)^2} \quad \Rightarrow \quad s = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\mathrm{d}y}{\mathrm{d}\theta}\right)^2 + \left(\frac{\mathrm{d}x}{\mathrm{d}\theta}\right)^2}\,\mathrm{d}\theta.$$

These formulae can be extended to find the curved surface of area of a solid of revolution S. Consider the curve AB rotated about the x-axis to form a thin shell

$$\delta S \approx 2\pi y\delta x\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}.$$

Summing these elements in the limit as $\delta x \to 0$ yields

$$S = 2\pi\int_{x_1}^{x_2} y\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\,\mathrm{d}x.$$

8 First Moments & Centres of Gravity

Given a system of masses, it is possible to define a point at which the entire system's mass can be assumed to be concentrated. This point is called the **centre of mass**. Because of its definition, a body made of a single mass $m = \sum_i^n m_i$ concentrated at the centre of mass will behave like the system which respect to the moment along the axis

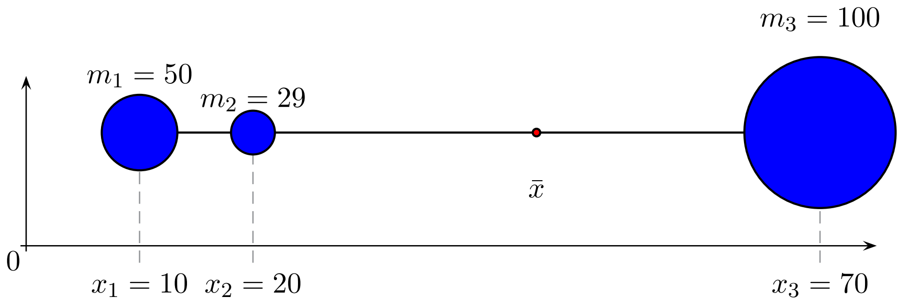
$$x_{\mathrm{cm}} m = \sum_{i=1}^{n} x_i m_i.$$

The sum of the individual moments due to each mass has to be equal to the moment of the equivalent global mass concentration at the centre of mass. Therefore

$$x_{\mathrm{cm}} = \frac{\sum_{i=1}^n x_i m_i}{m} = \frac{\sum_{i=1}^n x_i m_i}{\sum_i^n m_i}.$$

In the context of a constant gravitational field, the centre of mass is coincident with the centre of gravity.

Example 8.1. *For the system of masses illustrated where is the centre of mass?*



$$\begin{aligned}\bar{x} &= \frac{\sum_{i=1}^{3} x_i m_i}{\sum_{i=1}^{3} m_i} \\ &= \frac{10 \cdot 50 + 20 \cdot 29 + 70 \cdot 100}{50 + 29 + 100} \\ &= 45\end{aligned}$$

□

For a continuous body, C, of total mass m, the problem of determining the centre of mass can be reduced to elemental form.

(i) Define a elemental mass δm.

(ii) The distance in the yz-plane to the mass is x, thus compute the elemental moment $\delta M \approx x\delta m$.

(iii) Thus the first moment of mass M of the body C about the yz-plane is given by

$$M = \int x \, \mathrm{d}m \quad \text{as} \quad \delta m \to 0.$$

The distance between the yz-plane and the centre of gravity is denoted by $\bar{x}$ and is given by

$$m\bar{x} = M \Rightarrow \bar{x} = \frac{\int x \, \mathrm{d}m}{m}.$$

8.1 Centroid of Volume

If a body C has volume V has uniform density, i.e. ρ =const., then

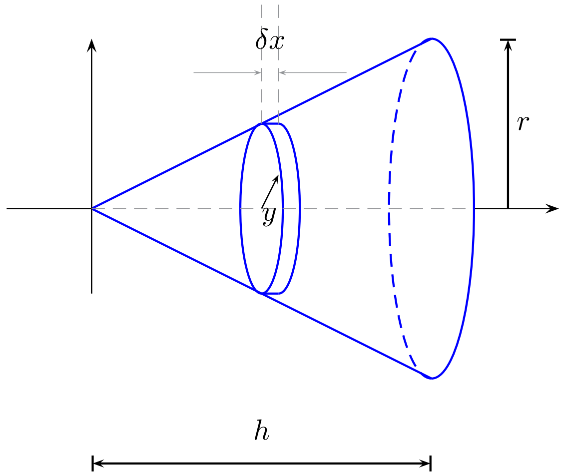
$$m = \rho V \quad \Rightarrow \quad \mathrm{d}m = \rho \, \mathrm{d}V.$$

Then the distance to the centre of mass $\bar{x}$ is given by

$$\bar{x} = \frac{1}{\rho V} \int x\rho \, \mathrm{d}V = \frac{1}{V} \int x \, \mathrm{d}V.$$

The **centroid of volume** is simply the first moment of volume divided by the volume of the body. If the density is constant then the centroid of volume will always be equal to the centre of gravity.

Example 8.2 (Solid of Revolution)**.** *Find the centre of gravity of a cone of uniform density ρ with radius r, height h.*



By symmetry the centre of gravity will be on the x-axis, thus $\bar{y} = \bar{z} = 0$. A suitable volume element is $\delta V \approx \pi y^2 \delta x$, where $y = rx/h$. Thus the first moment of volume about the x-axis is $\delta M_V^x \approx x\delta V \approx \pi x y^2 \delta x$. Therefore

$$\begin{aligned} M_V^x &= \pi \int_0^h x y^2 \, \mathrm{d}x \\ &= \pi \int_0^h x \left(rx/h\right)^2 \, \mathrm{d}x \\ &= \frac{\pi r^2}{h^2} \left[\frac{x^4}{4}\right]_0^h \\ &= \frac{\pi r^2 h^2}{4}. \end{aligned}$$

The volume is given by

$$\begin{aligned} V &= \pi \int_0^h \left(rx/h\right)^2 \, \mathrm{d}x \\ &= \frac{\pi r^2 h}{3}. \end{aligned}$$

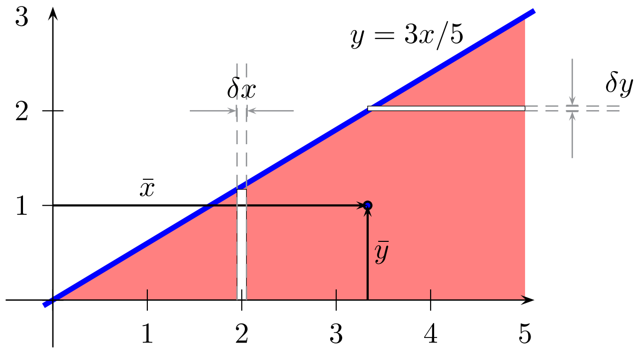
Thus $\bar{x} = \dfrac{3h}{4}$. □

8.2 Centroid of a Surface

Similarly, the **centroid of a surface** A is equal to the first moment of the area divided by the area, i.e.

$$\bar{x} = \frac{1}{A} \int x \, \mathrm{d}A.$$

Example 8.3. *Find the centroid of the triangular laminar bounded by the x-axis and the lines $y = 3x/5$ and $x = 5$.*



We note that the area of this triangle is $A = 15/2$ and that $\bar{x}$ is the perpendicular distance from the centroid and the yz-plane. A suitable area element is $\delta A \approx y\delta x$. Then the first moment of area about the yz-plane is $\delta M_A^x \approx x\delta A \approx xy\delta x$. Hence

$$\begin{aligned} M_A^x &= \int_0^5 xy\,\mathrm{d}x \\ &= \frac{3}{5}\int_0^5 x^2\,\mathrm{d}x \\ &= \frac{3}{5}\left[\frac{x^3}{3}\right]_0^5 \\ &= 25. \end{aligned}$$

Therefore $\bar{x} = 10/3$. Now to find $\bar{y}$ we note that the area element is $\delta A = (5 - x)\,\delta y = 5\,(1 - y/3)\,\delta y$, so the mass element is $5\rho\,(1 - y/3)\,\delta y$. So in the limit as $\delta x \to 0$

$$\begin{aligned} M_A^y &= 5\int_0^3 (1 - y/3)\,y\,\mathrm{d}y \\ &= 5\int_0^3 \left(y - y^2/3\right)\,\mathrm{d}y \\ &= 5\left[\frac{y^2}{2} - \frac{y^3}{9}\right]_0^3 \\ &= 15/2. \end{aligned}$$

So $\bar{y} = 1$. □

9 Second Moments

The kinetic energy of a system of global mass m, whose points move with a velocity v is given by

$$h = \frac{1}{2}\sum_{i=1}^{n} m_i v^2 = \frac{1}{2}mv^2.$$

In the case of a rotating rigid system this formulation has to be re-adapted: the velocity of each point of the system is a function of the angular velocity ω (given in radians per second) and the distance r from the axis of rotation, thus $v = \omega r$. Hence the kinetic energy is given by

$$h = \frac{1}{2}\omega^2\sum_{i=1}^{n} m_i r_i^2.$$

Let $I = \sum_{i=1}^{n} m_i r_i^2$ be the second moment of mass, thus

$$h = \frac{1}{2}\omega^2 I.$$

Thus the form of the kinetic energy is similar to that for linear velocity, where angular velocity corresponds to linear velocity, i.e. $v \leftrightarrow \omega$, and the **second moment of mass** I is equivalent to the inertial mass m in the case of rotating systems, i.e. $I \leftrightarrow m$. It gives a measure of a system's resistance to rotational accelerations about a given axis, i.e. it's moment of inertia.

For a continuous body C of total mass m, to find the moment of inertia we define a small element of mass δm at a distance r from the axis of rotation, thus $\delta I \approx r^2 \delta m$. Then in the limit as $\delta m \to 0$

$$I = \int_C r^2 \, \mathrm{d}m.$$

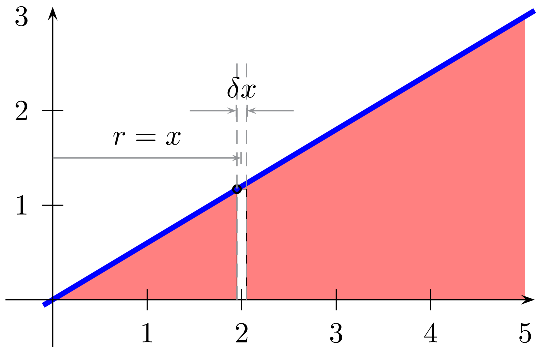
Since the second moment of mass is taken about an axis, the choice of elemental mass is influenced by the axis.

9.1 Second Moment of a Volume

If the body has $\rho =$ const. and volume V, then $m = \rho V$ and $\mathrm{d}m = \rho \, \mathrm{d}V$. Thus $I = \rho \int r^2 \, \mathrm{d}V$. The **second moment of volume** is given by

$$I_V = \int r^2 \, \mathrm{d}V.$$

Example 9.1 (Triangles)**.** *Find the moment of inertia of the triangular lamina, with uniform surface density ρ, about the y-axis.*



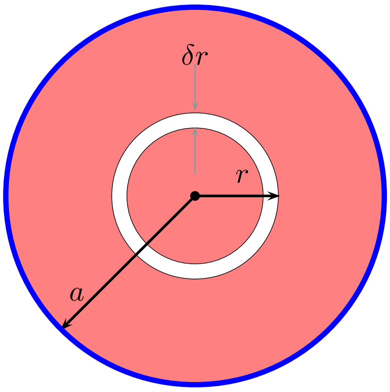
Choose a mass element that is equidistant from the y-axis: a vertical element of width δx and height y is ideal. Thus $\delta m \approx \rho y \delta x$. The distance of this element to the y-axis is simply x. Thus $\delta I_y \approx x^2 \delta m \approx \rho x^2 y \delta x$

$$\begin{aligned} I_y &= \frac{3\rho}{5} \int_0^5 x^3 \, \mathrm{d}x \\ &= \frac{3\rho}{5} \left[\frac{x^4}{4} \right]_0^5 \\ &= \frac{375\rho}{4}. \end{aligned}$$

□

First moments are taken about a plane, second moments are taken about an axis: this dictates the selection of an appropriate area.

Example 9.2 (Wheels). *Find the moment of inertia of the circular lamina of radius a and surface density ρ about its axis.*



Again, we choose an element which is equidistant from the axis at all points, so we choose an annulus of width δr. the area of an annulus is given by $\delta A = \pi\left(r^2 - (r-\delta r)^2\right) \approx 2\pi r\delta r$. Hence

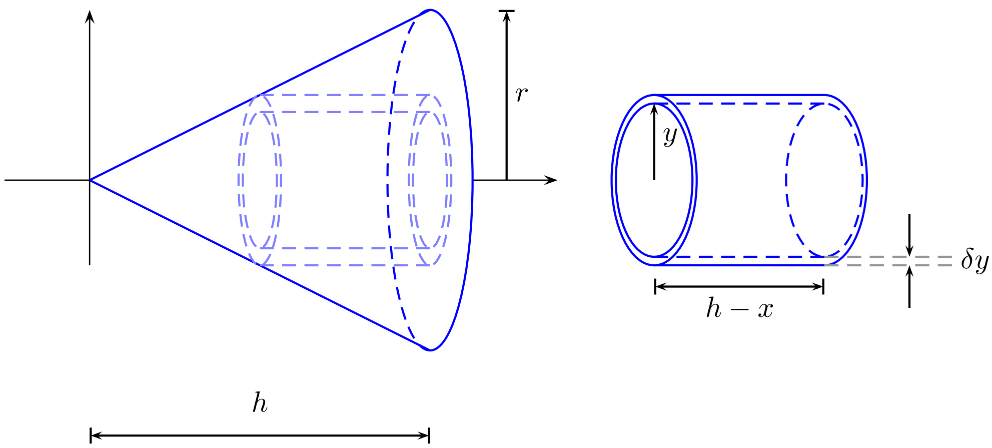
$$\begin{aligned}\delta I_0 &= r^2 \delta A \\ &= 2\pi\rho r^3 \delta r.\end{aligned}$$

Therefore

$$\begin{aligned}I_0 &= 2\pi\rho \int_0^a r^3 \,\mathrm{d}r \\ &= 2\pi\rho\left[\frac{r^4}{4}\right]_0^a \\ &= \frac{\pi\rho a^4}{2}.\end{aligned}$$

□

Example 9.3 (Moment of Inertia of a Solid of Revolution). *Firstly find the moment of inertia about O_x by choosing a mass element δm equidistant from O_x. Thus let the mass element be a cylindrical shell with axis along O_x, radius y and radial thickness δy.*



Thus

$$\delta m \approx 2\pi\rho y\,(h-x)\,\delta y.$$

Thus the moment of inertial about O_x is

$$\begin{aligned}\delta I_x &\approx y^2\delta m\\ &\approx 2\pi\rho y^3\,(h-x)\,\delta y\\ &= 2\pi\rho y^3\left(h-\frac{hy}{r}\right)\delta y.\end{aligned}$$

Thus

$$\begin{aligned}I_x &= 2\pi\rho h\int_0^r y^3-\frac{y^4}{r}\,\mathrm{d}y\\ &= 2\pi\rho h\left[\frac{y^4}{4}-\frac{y^5}{5r}\right]_0^r\\ &= \frac{\pi\rho h r^4}{10}.\end{aligned}$$

□

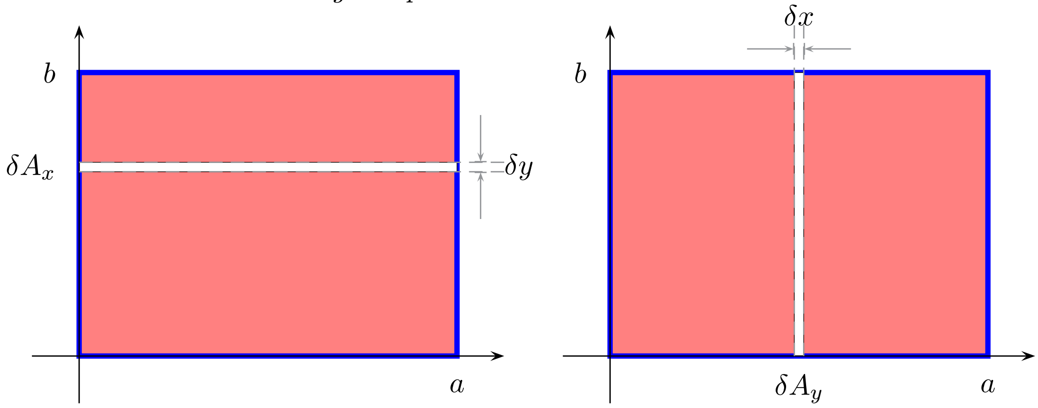
9.2 Second Moment of an Area

A related concept, which is applied to surfaces and sections, is the **second moment of area**. The second moment of area is defined as

$$I_A=\int r^2\,\mathrm{d}A.$$

The second moment of area of a section about an axis is an important concept in the study of elasticity as it gives a measure of the resistance of a section to bending along the axis. All results obtained for the second moment of mass for thin plates can be directly extended to this case by replacing the elemental mass with the elemental area.

Example 9.4. *Consider a rectangular plate.*



Then about the O_x plane the element area is given by $\delta A_x \approx a\delta y$. Thus $\delta I_{Ax} = y^2\delta A_x \approx ay^2\delta y$. Thus summing all incremental elements up in the limit as $\delta y \to 0$ gives

$$\begin{aligned}I_{Ax} &= a\int_0^b y^2\,\mathrm{d}y\\ &= \frac{ab^3}{3}.\end{aligned}$$

Similarly about the O_y plane the element area is given by $\delta A_y \approx b\delta x$. Thus $\delta I_{Ay} = x^2 \delta A_y \approx bx^2 \delta x$. Thus

$$\begin{aligned} I_{Ay} &= b \int_0^a x^2 \, \mathrm{d}x \\ &= \frac{a^3 b}{3}. \end{aligned}$$ □

9.3 Radius of Gyration

We have seen that it is possible to model the first moment due to a distributed mass as a point load acting through the centre of gravity of a body. Similarly it is possible to model a body rotating about an axis as a cylindrical shell of mass m equal to the total mass of the system, rotating about the same axis with the same angular speed. The radius of this cylindrical shell is known as the **radius of gyration**, denoted by k or R_g, and is obtained by equating the kinetic energy of the original body with the kinetic energy of the cylindrical shell. Thus the radius of gyration of a mass is given by

$$mk^2 = I \quad \Rightarrow k = \sqrt{\frac{I}{m}}.$$

An equivalent definition is the radius of gyration k of an area about an axis, i.e. about the x-axis as

$$Ak_x^2 = I_x \quad \Rightarrow k_x = \sqrt{\frac{I_x}{A}}.$$

Often there is no element equidistant to the axis about which we wish to find a second moment. In such cases we can apply the parallel axis theorem.

9.4 Parallel & Perpendicular Axis Theorems

If I_{AA} is the moment of inertia of a body of mass m about an axis AA through the centre of gravity of a body and I_{BB} the moment of inertia of the body through a parallel axis BB, then

$$I_{BB} = I_{AA} + mL^2$$

where L is the distance between AA and BB. The same result holds for second moments of area as

$$I_{BB} = I_{AA} + AL^2$$

where L is, again, the distance between AA and BB.

The same result holds for other second moments, for example for the radius of gyration

$$k_{AA}^2 = k_{BB}^2 + L^2.$$

For perpendicular axes

$$I_{O_z} = I_{O_x} + I_{O_y} \quad \text{and} \quad k_{O_z}^2 = k_{O_x}^2 + k_{O_y}^2$$

however in this case the result only holds for areas.

Example 9.5 (Rectangular Plares Revisited)**.** *We have seen that about O_x*

$$\begin{aligned} I_{Ax} &= a \int_0^b y^2 \, \mathrm{d}y \\ &= \frac{ab^3}{3}. \end{aligned}$$

So along the centre of gravity

$$\begin{aligned} I_{Gx} &= I_{Ax} - AL^2 \\ &= \frac{ab^3}{3} - ab\left(\frac{b}{2}\right)^2 \\ &= \frac{ab^3}{12}. \end{aligned}$$

Thus it is much easier for a section to bend along the centre of gravity. □

10 Techniques of Integration

10.1 Completing the Square

By completing the square, we can turn integrals into standard forms. For a given quadratic equation $ax^2 + bx + c$ we may divide through by a to get $x^2 + \frac{b}{a}x + \frac{c}{a}$.

As $(x+d)^2 = x^2 + 2dx + d^2$ then we express the quadratic as

$$\begin{aligned} x^2 + \frac{b}{a}x + \frac{c}{a} &= \left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + \frac{c}{a} - \frac{b^2}{4a^2} \\ &= \left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a^2}. \end{aligned}$$

If we let $z = x + \frac{b}{2a}$ and $\alpha = \sqrt{\frac{4ac - b^2}{4a^2}}$ then we can express the quadratic in the form $z^2 \pm \alpha^2$.

Example 10.1. *Find the indefinite integral*

$$\int \frac{1}{-2x^2 - 20x + 60}\,\mathrm{d}x.$$

At first glance the integral does not appear to be in any of our standard forms. However, by means of simple algebra the denominator may be rearranged to

$$\begin{aligned} \int \frac{1}{-2x^2 - 20x + 60}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{30 - (x^2 + 10x)}\,\mathrm{d}x \\ &= \frac{1}{2}\int \frac{1}{30 + 25 - (x^2 + 10x + 25)}\,\mathrm{d}x \\ &= \frac{1}{2}\int \frac{1}{55 - (x+5)^2}\,\mathrm{d}x. \end{aligned}$$

Thus the integral is now in the form

$$\int \frac{1}{a^2 - x^2}\,\mathrm{d}x$$

where $a = \sqrt{55}$. Thus

$$\begin{aligned} \int \frac{1}{-2x^2 - 20x + 60}\,\mathrm{d}x &= \frac{1}{2}\int \frac{1}{55 - (x+5)^2}\,\mathrm{d}x \\ &= \frac{1}{4\sqrt{55}} \ln\left|\frac{\sqrt{55} + 5 + x}{\sqrt{55} - 5 - x}\right| + c. \end{aligned}$$ □

10.2 Rational Functions & Partial Fractions

In many applications we need to integrate rational functions, that is functions of the form

$$f(x) = \frac{h(x)}{g(x)}$$

where $h(x)$ and $g(x)$ are polynomials. Rational functions can be expressed as a series of partial fractions which can then be evaluated by our standard forms

Example 10.2. *Evaluate*

$$\int f(x)\,\mathrm{d}x = \int \frac{x+7}{x^2-7x+10}\,\mathrm{d}x.$$

Although the integral does not look like one of our standard forms, it can be expressed as a partial fraction. First we note that

$$x^2 - 7x + 10 = (x-5)(x-2).$$

Then, as the numerator is a polynomial of degree one and the denominator a polynomial of degree two, assume that the function can be expressed in the form

$$\begin{aligned} f(x) &= \frac{a}{x-5} + \frac{b}{x-2} \\ &= \frac{a(x-2)}{(x-5)(x-2)} + \frac{b(x-5)}{(x-5)(x-2)} \\ &= \frac{a(x-2)+b(x-5)}{(x-5)(x-2)} \\ &= \frac{x(a+b)-(5a+2b)}{(x-5)(x-2)}. \end{aligned}$$

Thus, on equating powers of x we have a pair of linear simultaneous equations, i.e.

$$1 = a + b \quad \textit{and} \quad 7 = -2a - 5b.$$

Let $a = 1 - b$, then $7 = -2(1-b) - 5b = -9 = 3b$, thus $b = -3$ and so $a = 4$. Therefore

$$f(x) = \frac{4}{x-5} - \frac{3}{x-2}.$$

So that the integral is

$$\begin{aligned} \int f(x)\,\mathrm{d}x &= 4\int \frac{1}{x-5}\,\mathrm{d}x - 3\int \frac{1}{x-2}\,\mathrm{d}x \\ &= 4\ln|x-5| - 3\ln|x-2| + c \\ &= \ln\left|\frac{(x-5)^4}{(x-2)^3}\right| + c. \end{aligned}$$

□

10.3 Change of Variables

An important technique is to change the variables over which we are integrating. The key to the technique is the correct choice of substitution. Experience and practice play a large part in the ease at which the process applied.

Example 10.3. *Consider the integral*

$$\int \frac{\mathrm{d}x}{x\left(1+x^2\right)}.$$

The substitution $u = x^2$ *implies* $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ *hence* $\mathrm{d}u = 2x\,\mathrm{d}x$, *i.e.* $\mathrm{d}x = \frac{\mathrm{d}u}{2\sqrt{u}}$. *Thus*

$$\int \frac{\mathrm{d}x}{x\left(1+x^2\right)} = \frac{1}{2}\int \frac{\mathrm{d}u}{u\left(1+u\right)}.$$

The integral can now be solved using partial fractions

$$\begin{aligned}
\frac{1}{2}\int \frac{\mathrm{d}u}{u\left(1+u\right)} &= \frac{1}{2}\int \frac{\mathrm{d}u}{u} - \frac{1}{2}\int \frac{\mathrm{d}u}{1+u} \\
&= \frac{1}{2}\left(\ln u - \ln\left(1+u\right)\right) + c \\
&= \frac{1}{2}\ln\frac{u}{1+u} + c \\
&= \ln\sqrt{\frac{u}{1+u}} + c.
\end{aligned}$$

Back-substituting $u = x^2$ *gives the solution*

$$\int \frac{\mathrm{d}x}{x\left(1+x^2\right)} = \ln\frac{|x|}{\sqrt{1+x^2}}.$$

□

There are many different choices of substitution, for example

Example 10.4. *Consider the integral*

$$\int \frac{\mathrm{d}x}{x\left(1+x^2\right)}.$$

The substitution $u = 1 + x^2$ *implies* $\frac{\mathrm{d}u}{\mathrm{d}x} = 2x$ *hence* $\mathrm{d}u = 2x\,\mathrm{d}x$, *i.e.* $\mathrm{d}x = \frac{\mathrm{d}u}{2\sqrt{u-1}}$. *Thus*

$$\begin{aligned}
\int \frac{\mathrm{d}x}{x\left(1+x^2\right)} &= \frac{1}{2}\int \frac{\mathrm{d}u}{u\left(u-1\right)} \\
&= \frac{1}{2}\int \frac{\mathrm{d}u}{u-1} - \frac{1}{2}\int \frac{\mathrm{d}u}{u} \\
&= \ln\sqrt{\frac{u-1}{u}} + c.
\end{aligned}$$

Back-substituting $u = 1 + x^2$ *gives the correct solution.* □

The two examples highlight the value of experience in selecting a substitution. The following is a selection of important cases with their suggested substitutions

Example 10.5. *Find the indefinite integral*

$$\int x\sqrt{3x-1}\,\mathrm{d}x.$$

Expression in integral	Suggested substitution
$\sqrt{ax+b}$	$ax+b=u^2$
$x\sqrt{ax+b}$	$x=1/u$
$\sqrt{a^2-x^2}$	$x=a\sin\theta$ or $x=a\tanh u$
$\sqrt{x^2-a^2}$	$x=a\sec^2\theta$ or $x=a\cosh u$
$\sqrt{x^2+a^2}$	$x=a\tan\theta$ or $x=a\sinh u$
x^2+a^2	$x=a\tan\theta$

Let $u=\sqrt{3x-1}$*, that is* $x=\left(u^2+1\right)/3$*, so that* $\dfrac{\mathrm{d}x}{\mathrm{d}u}=\dfrac{2u}{3}$*. Thus the integral becomes*

$$\begin{aligned}\int x\sqrt{3x-1}\,\mathrm{d}x &= \int u\frac{u^2+1}{3}\frac{2u}{3}\,\mathrm{d}u\\ &= \frac{2}{9}\int u^4+u^2\,\mathrm{d}u\\ &= \frac{2}{9}\left(\frac{u^5}{5}+\frac{u^3}{3}\right)+c\\ &= \frac{2u^3}{135}\left(3u^2+5\right)+c.\end{aligned}$$

Back-substitution yields the solution in terms of x

$$\int x\sqrt{3x-1}\,\mathrm{d}x=\frac{2}{135}(3x-1)^{3/2}(9x+2)+c. \qquad \square$$

Example 10.6. *Find the indefinite integral*

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x.$$

Let $x=a\sin\theta$ *then* $\mathrm{d}x=a\cos\theta\,\mathrm{d}\theta$ *and hence*

$$\begin{aligned}\int\sqrt{a^2-x^2}\,\mathrm{d}x &= \int\sqrt{a^2-a^2\sin^2\theta}\,a\cos\theta\,\mathrm{d}\theta\\ &= a\int\sqrt{a^2\left(1-\sin^2\theta\right)}\cos\theta\,\mathrm{d}\theta\\ &= a^2\int\cos^2\theta\,\mathrm{d}\theta\\ &= \frac{a^2}{2}\int(1+\cos 2\theta)\,\mathrm{d}\theta\\ &= \frac{a^2}{2}\left(\theta+\frac{1}{2}\sin 2\theta\right)+c.\end{aligned}$$

Back substitution requires the use of the trigonometric identities

$$\sin\theta=\frac{x}{a} \quad\Rightarrow\quad \cos\theta=\frac{\sqrt{a^2-x^2}}{a} \quad \textit{hence} \quad \frac{1}{2}\sin 2\theta=\sin\theta\cos\theta=\frac{x\sqrt{a^2-x^2}}{a^2}.$$

Hence

$$\int\sqrt{a^2-x^2}\,\mathrm{d}x=\frac{a^2}{2}\sin^{-1}\frac{x}{a}+\frac{x}{2}\sqrt{a^2-x^2}+c. \qquad \square$$

The example demonstrates the importance of the trigonometric identities in integration (and calculus in general). You will frequently be required to manipulate trigonometric expressions into a form which can easily be integral and as such it is essential that the following identities are known:

$$\sin^2\theta + \cos^2\theta = 1,$$

$$\begin{aligned}\sin(\alpha \pm \beta) &= \sin\alpha\cos\beta \pm \cos\alpha\sin\beta,\\ \cos(\alpha \pm \beta) &= \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.\end{aligned}$$

Hence we have the double angle formulae

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta,\\ \cos 2\theta &= \cos^2\theta - \sin^2\theta,\\ \tan 2\theta &= \frac{2\tan\theta}{1-\tan^2\theta}\end{aligned}$$

and the following useful identities

$$\cos^2\theta = \frac{1}{2}(1+\cos\theta) \quad \text{and} \quad \sin^2\theta = \frac{1}{2}(1-\cos\theta).$$

Definite integrals can be solved using by substitution but care is need to ensure the limits of integration correspond to the correct limits in the new variable, as the following example shows

Example 10.7. *Evaluate the definite integral*

$$\int_{1/2}^{3} x\sqrt{2x+3}\,\mathrm{d}x.$$

Let $u = 2x+3$ *then when* $x = 1/2$ *then* $u = 4$ *and when* $x = 3$ *then* $u = 9$*. Also* $\mathrm{d}u = 2\mathrm{d}x$*, hence*

$$\begin{aligned}\int_{1/2}^{3} x\sqrt{2x+3}\,\mathrm{d}x &= \frac{1}{4}\int_4^9 (u-3)\sqrt{u}\,\mathrm{d}u\\ &= \frac{1}{4}\int_4^9 \left(u^{3/2} - 3u^{1/2}\right)\mathrm{d}u\\ &= \frac{1}{4}\left[\frac{2u^{5/2}}{5} - 2u^{3/2}\right]_4^9\\ &= \frac{116}{10}.\end{aligned}$$

□

The substitution $u = \tan x$ is useful in evaluating quotients involving the square of trigonometric functions

$$\int \frac{\mathrm{d}x}{a + b\cos^2 x + c\sin^2 x}.$$

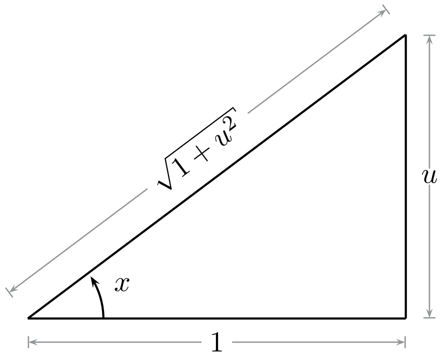
We need to find expressions for $\dfrac{\mathrm{d}x}{\mathrm{d}u}$, $\cos^2 x$ and $\sin^2 x$

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \sec^2 x = 1 + \tan^2 x = 1 + u^2 \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{1}{1+u^2}.$$

Thus, from $\sec^2 x = 1 + u^2$ so $\cos^2 x = \dfrac{1}{1+u^2}$ and $\sin^2 x = \dfrac{u^2}{1+u^2}$. Hence

$$\int \frac{\mathrm{d}x}{a + b\cos^2 x + c\sin^2 x} = \int \frac{\mathrm{d}u}{u^2(a+b) + (a+c)}$$

which can be expressed in standard form and then solved.



Example 10.8. *Find the definite integral*

$$\int \frac{\mathrm{d}x}{7 + \cos^2 x}.$$

From the substitution $u = \tan x$ *then the integral is*

$$\begin{aligned}\int \frac{\mathrm{d}x}{7+\cos^2 x} &= \int \frac{\mathrm{d}u}{8u^2+7} \\ &= \frac{1}{8}\int \frac{\mathrm{d}u}{u^2 + \left(\sqrt{7/8}\right)^2}\end{aligned}$$

which integrates to give

$$\int \frac{\mathrm{d}x}{7+\cos^2 x} = \frac{1}{\sqrt{56}} \tan^{-1} \sqrt{7/8}\, u + c.$$

Finally since $u = \tan x$, *then back-substituting gives*

$$\int \frac{\mathrm{d}x}{7+\cos^2 x} = \frac{1}{\sqrt{56}} \tan^{-1} \left(\sqrt{8/7}\tan x\right) + c.$$

□

The second substitution we will consider is the $u = \tan x/2$. Typically this substitution is used for integrals of the form

$$\int \frac{\mathrm{d}x}{a + b\cos x + c\sin x}.$$

Thus

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1 + \tan^2\frac{x}{2}\right) = \frac{1+u^2}{2} \quad \Rightarrow \quad \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{2}{1+u^2}.$$

From $\sec^2\dfrac{x}{2} = \dfrac{1+u^2}{2}$ so $\cos\dfrac{x}{2} = \dfrac{1}{\sqrt{1+u^2}}$ and $\sin\dfrac{x}{2} = \dfrac{u}{\sqrt{1+u^2}}$. Hence, by the trigonometric identities

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = \frac{2u}{1+u^2} \quad \text{and} \quad \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-u^2}{1+u^2}.$$

Therefore

$$\int \frac{\mathrm{d}x}{a + b\cos x + c\sin x} = 2\int \frac{\mathrm{d}u}{(a-b)\,u^2 + 2cu + (a+b)}.$$

Example 10.9. *Find the indefinite integral*

$$\int \frac{\mathrm{d}x}{1+\sin x}.$$

From the substitution $u = \tan\dfrac{x}{2}$ *then* $\sin x = \dfrac{2u}{1+u^2}$ *and* $\dfrac{\mathrm{d}x}{\mathrm{d}u} = \dfrac{2}{1+u^2}$ *so that*

$$\begin{aligned}
\int \frac{\mathrm{d}x}{1+\sin x} &= \int \frac{2/\left(1+u^2\right)}{1+2u/\left(1+u^2\right)}\mathrm{d}u \\
&= \int \frac{2\mathrm{d}u}{1+2u+u^2} \\
&= \int \frac{2\mathrm{d}u}{(1+u)^2} \\
&= -\frac{2}{1+u} + c \\
&= -\frac{2}{1+\tan x/2} + c.
\end{aligned}$$

□

10.4 Integrals Containing Derivatives of Functions

Although not listed in the table of standard integrals, integrals of the form

$$\int f'(x)\,f(x)\,\mathrm{d}x \quad \text{and} \quad \int \frac{f'(x)}{f(x)}\,\mathrm{d}x$$

can easily be solved by inspection as

$$\int f'(x)\,f(x)\,\mathrm{d}x = \frac{1}{2}f^2(x) + c \quad \text{and} \quad \int \frac{f'(x)}{f(x)}\mathrm{d}x = \ln|f(x)| + c.$$

To derive the results, let $f(x) = z$, then $f'(x)\,\mathrm{d}x = \mathrm{d}z$ and the equations are in standard form.

10.5 Integration by Parts

It is not always possible to express a product in the form $f'(x)\,f(x)$, so an alternative method of integration is required. If $u = u(x)$ and $v = v(x)$, then the differential of the product of the two functions is given by the product rule as

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}.$$

Integrating both sides of this expression yields

$$uv = \int \left(u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x} \right) \mathrm{d}x.$$

We can rearrange as

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\,\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\,\mathrm{d}x.$$

At first glance it may appear that no progress has been made - we simply have replaced one integral with another. However the correct choice of u and v the integral should be more easily solved than the original integral.

Some problems require successive applications of integration by parts, or the use of another formula, for example

Example 10.10. *Find*

$$\int e^x \sin x\,\mathrm{d}x.$$

By parts, let

$$\begin{aligned} u = e^x \quad &\Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = e^x \\ \frac{\mathrm{d}v}{\mathrm{d}x} = \sin x \quad &\Rightarrow \quad v = -\cos x. \end{aligned}$$

Thus

$$\int e^x \sin x\,\mathrm{d}x = -e^x \cos x + \int e^x \cos x\,\mathrm{d}x.$$

If we now apply integration by parts to

$$\int e^x \cos x\,\mathrm{d}x$$

with

$$\begin{aligned} u = e^x \quad &\Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = e^x \\ \frac{\mathrm{d}v}{\mathrm{d}x} = \cos x \quad &\Rightarrow \quad v = \sin x. \end{aligned}$$

Then

$$\int e^x \cos x\,\mathrm{d}x = e^x \sin x + \int e^x \sin x\,\mathrm{d}x.$$

On combining and simplifying the two expressions we find

$$\int e^x \sin x\,\mathrm{d}x = \frac{1}{2}e^x\left(\sin x - \cos x\right) + c.$$ □

A useful particular case of integration by parts occurs when we part $v = x$ into the integration by parts formula. We are multiplying the integrand by 1 and applying integration by parts to the product. An important application of this technique is applied to the natural logarithm $\ln x$.

Example 10.11. *Find*

$$\int \ln x \, \mathrm{d}x.$$

Let

$$\begin{aligned} u = \ln x \quad &\Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{x} \\ \frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad &\Rightarrow \quad v = x \end{aligned}$$

hence

$$\begin{aligned} \int \ln x \, \mathrm{d}x &= x \ln x - \int 1 \mathrm{d}x \\ &= x (\ln x - 1) + c. \end{aligned}$$

□

Example 10.12. *Find*

$$\int \sqrt{a^2 - x^2} \, \mathrm{d}x.$$

Let

$$\begin{aligned} u = \sqrt{a^2 - x^2} \quad &\Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{-x}{\sqrt{a^2 - x^2}} \\ \frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad &\Rightarrow \quad v = x \end{aligned}$$

hence

$$\int \sqrt{a^2 - x^2} \, \mathrm{d}x = x\sqrt{a^2 - x^2} + \int \frac{x^2}{\sqrt{a^2 - x^2}} \mathrm{d}x.$$

Now note that

$$\begin{aligned} \int \sqrt{a^2 - x^2} \, \mathrm{d}x &= \int \frac{a^2 - x^2}{\sqrt{a^2 - x^2}} \, \mathrm{d}x \\ &= \int \frac{a^2}{\sqrt{a^2 - x^2}} \, \mathrm{d}x - \int \frac{x^2}{\sqrt{a^2 - x^2}} \, \mathrm{d}x \\ &= a^2 \sin^{-1} \frac{x}{a} - \int \frac{x^2}{\sqrt{a^2 - x^2}} \, \mathrm{d}x. \end{aligned}$$

Thus combining the two expression and simplifying yields

$$\int \sqrt{a^2 - x^2} \, \mathrm{d}x = \frac{a^2}{2} \sin^{-1} \frac{x}{a} + \frac{x}{2}\sqrt{a^2 - x^2} + c.$$

□

When using integration by parts to evaluate a definite integral the formula is now given by

$$\int_a^b u \frac{\mathrm{d}v}{\mathrm{d}x} \, \mathrm{d}x = [uv]_a^b - \int_a^b v \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

It may be the case that the integral on the right hand side can be evaluate by a change of variable, in which case the limits of integration would need to be changed and would be different from those used to evaluate the uv term.

Example 10.13. *Evaluate the definite integral*

$$\int_0^{1/2} \cos^{-1} x \, \mathrm{d}x.$$

This is a special case of integration by parts

$$u = \cos^{-1} x \quad \Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}x} = -\frac{1}{\sqrt{1-x^2}},$$
$$\frac{\mathrm{d}v}{\mathrm{d}x} = 1 \quad \Rightarrow \quad v = x.$$

Integration by parts gives

$$\int_0^{1/2} \cos^{-1} x \, \mathrm{d}x = \left[x \cos^{-1} x\right]_0^{1/2} + \int_0^{1/2} \frac{x}{\sqrt{1-x^2}} \, \mathrm{d}x$$

we can evaluate the integral on the right hand side via the substitution $u = 1 - x^2$ *so that* $\dfrac{du}{dx} = -2x$ *hence* $x \, \mathrm{d}x = -\dfrac{\mathrm{d}u}{2}$. *The limits of integration change as when* $x = 0$ *then* $u = 1$ *and when* $x = 1/2$ *then* $u = 3/4$. *Thus the integral is now*

$$\begin{aligned}\int_0^{1/2} \cos^{-1} x \, \mathrm{d}x &= \left[x \cos^{-1} x\right]_0^{1/2} - \frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}} \, \mathrm{d}u \\ &= \frac{\pi}{6} - \left[\sqrt{u}\right]_1^{3/4} \\ &= \frac{\pi}{6} + 1 - \sqrt{\frac{3}{4}}.\end{aligned}$$

□

11 Reduction Formulae

Previously we have looked at the technique of integration by parts as a means of integrating products. The technique is reliant on reducing the original integral to the difference of a product and another integral of simpler form. Often the reduced integral is of the same form as the original integral.

Example 11.1. *Find*

$$\int x^3 e^x \, \mathrm{d}x.$$

Let $u = x^3$ *then* $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ *and* $v = e^x$ *then* $\dfrac{\mathrm{d}v}{\mathrm{d}x} = e^x$. *Thus*

$$\int x^3 e^x \, \mathrm{d}x = x^3 e^x - 3\int x^2 e^x \, \mathrm{d}x.$$

The second integral is in a similar form to the original integral, i.e. a polynomial term, x^3 *or* x^2, *multiplied by the exponential function* e^x. *Indeed, we may write*

$$\int x^n e^x \, \mathrm{d}x = x^n e^x - n\int x^{n-1} e^x \, \mathrm{d}x$$

and we may denote $I_n = \int x^n e^x \, \mathrm{d}x$ *so that* $I_n = x^n - nI_{n-1}$ *where* $I_n = x^{n-1}e^x - (n-1)\, I_{n-1}$ *and so on.* □

Reduction formulae can be written for many common integrals and can simplify the process of integration.

Example 11.2. *Express, in reduced form, the integral*

$$\int \cos^n \theta \, \mathrm{d}\theta.$$

Firstly let $\cos^n \theta = \cos\theta \cos^{n-1}\theta$*, then using integration by parts with*

$$\begin{aligned} u = \cos^{n-1}\theta \quad &\Rightarrow \quad \frac{\mathrm{d}u}{\mathrm{d}\theta} = -(n-1)\cos^{n-2}\theta\sin\theta \\ \frac{\mathrm{d}v}{\mathrm{d}\theta} = \cos\theta \quad &\Rightarrow \quad v = \sin\theta. \end{aligned}$$

Then

$$\begin{aligned} \int \cos^n\theta \, \mathrm{d}\theta &= \cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta\sin^2\theta \, \mathrm{d}\theta \\ &= \cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta\left(1-\cos^2\theta\right) \mathrm{d}\theta \\ &= \cos^{n-1}\theta\sin\theta + (n-1)\int \cos^{n-2}\theta \, \mathrm{d}\theta - \int \cos^n\theta \, \mathrm{d}\theta. \end{aligned}$$

Denoting $I_n = \int \cos^n\theta \, \mathrm{d}\theta$ *then, from the above equation*

$$I_n = \cos^{n-1}\theta\sin\theta + (n-1)\left(I_{n-2} - I_n\right).$$

Hence

$$I_n = \frac{1}{n}\left(\cos^{n-1}\theta\sin\theta + (n-1)\, I_{n-2}\right). \qquad \square$$

The following example shows the value of reduction formulae. Note that in the case of definite integrals reduction formulae can be used to find exact solutions.

Example 11.3. *Find the indefinite integral*

$$\int \cos^4\theta \, \mathrm{d}\theta.$$

From the previous example we know

$$\begin{aligned} I_4 &= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}I_2 \\ I_2 &= \frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}I_0 \\ I_0 &= \int \cos^0\theta \, \mathrm{d}\theta = \int \mathrm{d}\theta = \theta. \end{aligned}$$

Thus

$$\begin{aligned} \int \cos^4\theta \, \mathrm{d}\theta &= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\theta\right) + c \\ &= \frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{8}\cos\theta\sin\theta + \frac{3}{8}\theta + c. \end{aligned} \qquad \square$$

Example 11.4. *Evaluate the definite integral*

$$\int_0^{\pi/2} \cos^6\theta \,\mathrm{d}\theta.$$

Now in the derivation of the reduction formula we must evaluate the terms in the integral, thus we now have

$$I_n = \frac{1}{n}\left[\cos^{n-1}\theta\sin\theta\right]_0^{\pi/2} + \frac{(n-1)}{n}I_{n-2}.$$

We can see that for $n > 1$ the first part of the formula is zero when evaluated over the given range. Thus, in this case $I_n = \dfrac{n-1}{n}I_{n-2}$ for $n > 1$. If n is odd we will eventually be left with the case of $n = 1$, in which case

$$I_1 = \int_0^{\pi/2} \cos\theta \,\mathrm{d}\theta = 1.$$

If n is even we will be left with the limiting case $n = 0$, in which case

$$I_0 = \int_0^{\pi/2} \mathrm{d}\theta = \frac{\pi}{2}.$$

Therefore the integral can be evaluated as

$$I_6 = \frac{5}{6}I_4 = \frac{5}{6}\frac{3}{4}I_2 = \frac{5}{6}\frac{3}{4}\frac{1}{2}I_0 = \frac{5}{6}\frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{5\pi}{32}.$$ □

12 Infinite Integrals

Thus far we have assumed that all our functions are evaluated over a finite, bounded domain. However situations may arise where the limits of integration become infinite. Consider

$$\int_a^{\infty} f(x)\,\mathrm{d}x$$

then

$$\int_a^{\infty} f(x)\,\mathrm{d}x = \lim_{b\to\infty}\int_a^b f(x)\,\mathrm{d}x = \lim_{b\to\infty}\left[F(b)\right] - F(a).$$

A similar approach holds for integrals of the form

$$\int_{-\infty}^{b} f(x)\,\mathrm{d}x = \lim_{a\to-\infty}\int_a^b f(x)\,\mathrm{d}x = F(b) - \lim_{a\to-\infty}\left[F(a)\right].$$

Note that often it is not always possible to evaluate a function when its argument tends to infinity.

Example 12.1. *Evaluate the integral*

$$\int_0^{\infty} \frac{1}{(1+x^2)^2}\,\mathrm{d}x.$$

By substitution

$$x = \tan u \quad \textit{then} \quad \frac{\mathrm{d}x}{\mathrm{d}u} = \sec^2 u \quad\Rightarrow\quad \mathrm{d}x = \sec^2 u\,\mathrm{d}u$$

and the limits change as when

$$\begin{aligned} x \to \infty \quad &\Rightarrow \quad u = \frac{\pi}{2} \\ x = 0 \quad &\Rightarrow \quad u = 0. \end{aligned}$$

Hence

$$\begin{aligned} \int_0^\infty \frac{1}{(1+x^2)^2}\,\mathrm{d}x &= \int_0^{\pi/2} \frac{\sec^2 u\,\mathrm{d}u}{(1+\tan^2 u)^2} \\ &= \int_0^{\pi/2} \frac{\mathrm{d}u}{\sec^2 u} \\ &= \int_0^{\pi/2} \cos^2 u\,\mathrm{d}u \\ &= \frac{1}{2}\int_0^{\pi/2} (1+\cos 2u)\,\mathrm{d}x \\ &= \frac{1}{2}\left[u + \frac{1}{2}\sin 2u\right]_0^{\pi/2} \\ &= \frac{\pi}{4}. \end{aligned}$$

□

It may also be the case that the function itself tends to infinity at one of the limits of integration, say b. In this case we remove the infinite value from the summation by perturbing the limit b by a small parameter ϵ

$$\int_a^{b-\epsilon} f(x)\ \mathrm{d}x \quad \text{for} \quad \epsilon > 0.$$

Then we consider the limiting case as $\epsilon \to 0$

$$\int_a^{b-\epsilon} f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0}\int_a^{b-\epsilon} f(x)\ \mathrm{d}x = \lim_{\epsilon\to 0}\left[F(b-\epsilon)\right] - F(a).$$

Example 12.2. *Evaluate the definite integral*

$$\int_0^1 \ln x\,\mathrm{d}x$$

where $\ln 0$ *is infinite. Thus we let*

$$\begin{aligned} \int_{0+\epsilon}^1 \ln x\,\mathrm{d}x &= [x\ln x - x]_\epsilon^1 \\ &= \ln 1 - \epsilon\ln\epsilon + \epsilon - 1 \\ &= -\epsilon\ln\epsilon + \epsilon - 1 \end{aligned}$$

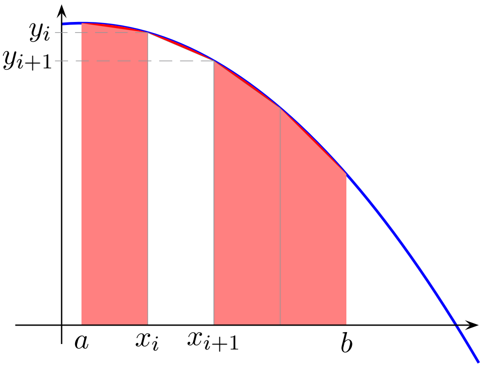
As $\lim_{\epsilon\to 0}[\epsilon\ln\epsilon] = 0$ *then* $\int_0^1 \ln x\,\mathrm{d}x = -1.$ □

13 Numerical Integration

In many real world applications no exact analytical solution to a definite integral is possible, thus an approximate solution to such problems must be found via numerical methods. Most numerical methods are based upon the division of the domain into a finite number of very small elements whose areas can be summed to give an approximate solution. The two most commonly applied methods are known as the trapezium and Simpson's Rule.

13.1 Trapezium Rule

Consider the area underneath the curve $f(x)$ between the limits $x = a$ and $x = b$.



Calculating the area of the i^{th}-strip $\mathrm{d}A_i$, with $d = x_{i+1} - x_i$, then

$$\begin{aligned}\mathrm{d}A_i &\approx dy_{i+1} + \frac{d}{2}(y_i - y_{i+1}) \\ &= \frac{d}{2}(y_i + y_{i+1}).\end{aligned}$$

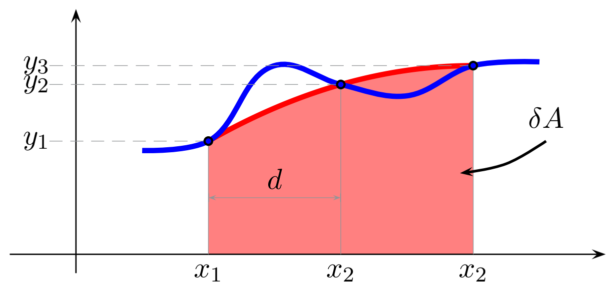
Summing all the individual areas gives

$$\begin{aligned}A &\approx \sum_{i=1}^{n} dA_i \\ &= \frac{d}{2}(y_1 + y_2) + \frac{d}{2}(y_1 + y_2) + \ldots + \frac{d}{2}(y_{n-1} + y_n) \\ &= \frac{d}{2}(y_1 + 2y_2 + 2y_3 + \ldots + 2y_{n-1} + y_n).\end{aligned}$$

Note that the ordinates must be evenly spaced, but that any number of elements can be used.

13.2 Simpson's Rule

The trapezium method using linear approximations, Simpson's method of numerical integration approximates the area under a curve by a series of quadratic functions passing through three points of the curve.



Consider the area represented by two strips of equal width d, with values x_1, x_2 and x_3 where $x_1 = x_2 - d$ and $x_3 = x_2 + d$. The function given by $y_1 = f(x_1) = f(x_1 - d)$ etc. Approximating the integrand as a quadratic function, the area can be approximated as

$$\int_{x_1}^{x_3} f(x)\,\mathrm{d}x \approx \int_{x_2-d}^{x_2+d} \left(ax^2 + bx + c\right)\,\mathrm{d}x.$$

On integrating and substituting in the values of y_1, y_2 and y_3, the area can be shown to be given by

$$dA_2 \approx \frac{d}{3}\left(y_1 + 4y_2 + y_3\right).$$

Summing all strips

$$\begin{aligned} A &\approx \sum_{i=1}^{n} dA_{2i} \\ &= \frac{d}{3}\left(y_1 + 4y_2 + y_3\right) + \frac{d}{3}\left(y_3 + 4y_4 + y_5\right) + \ldots + \frac{d}{3}\left(y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{d}{3}\left(y_1 + 4y_2 + 2y_3 + 4y_4 \ldots + 2y_{n-2} + 4y_{n-1} + y_n\right). \end{aligned}$$

Note that an even number of strips is required, i.e. $n-1$, as an odd number of ordinates, n, is required.

Example 13.1. *Use Simpson's rule to with five ordinates to find an approximation to*

$$A = \int_0^{\pi} \sqrt{\sin\theta}\, \mathrm{d}\theta.$$

Thus, when $n = 5$ *then* $d = \pi/4$*, so* $x_1 = 0$*,* $x_2 = \pi/4$*,* $x_3 = \pi/2$*,* $x_4 = 3\pi/4$ *and* $x_5 = \pi$*. Then by Simpson's rule*

$$\begin{aligned} A &\approx \frac{1}{3} \cdot \frac{\pi}{4}\left(\sqrt{\sin x_1} + 4\sqrt{\sin x_2} + 2\sqrt{\sin x_3} + 4\sqrt{\sin x_4} + \sqrt{\sin x_5}\right) \\ &= \frac{1}{3} \cdot \frac{\pi}{4}\left(0 + 4 \cdot 2^{-1/4} + 2 \cdot 1 + 4 \cdot 2^{-1/4} + 0\right) \\ &= 2.2848 \text{ to four decimal places.} \end{aligned}$$

□