Vectors Quiz Solutions 2011

1.

$\underline{b}\cdot\underline{a} = -1\times1 + 3\times4 + 5\times-5 = -1+12-25 = -14$

$\underline{c}\cdot\underline{a} = 1\times1 + 3\times0 + 2\times5 = 1+0+10 = 11$ 1

$(\underline{b}\cdot\underline{a})\underline{c} - (\underline{a}\cdot\underline{c})\underline{b} = 11(-1,+4,-5) - 14(1,0,2) = (-11,44,-55)-(14,0,-28)$

$= (-25, 44, -83)$ 1

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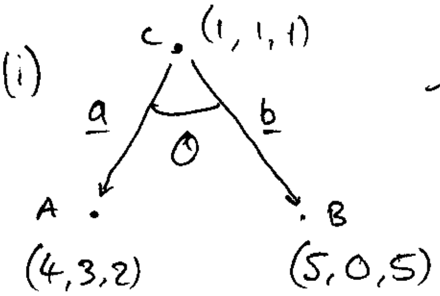
2. (i)



let $\underline{a} = \overrightarrow{CA} = (4,3,2)-(1,1,1) = (3,2,1)$

$\underline{b} = \overrightarrow{CB} = (5,0,5)-(1,1,1) = (4,-1,4)$ 1

$|\underline{a}| = \sqrt{3^2+2^2+1^2} = \sqrt{9+4+1} = \sqrt{14}$ 1

$|\underline{b}| = \sqrt{4^2+(-1)^2+4^2} = \sqrt{16+1+16} = \sqrt{33}$

$\underline{a}\cdot\underline{b} = |\underline{a}||\underline{b}|\cos\theta = \sqrt{14}\sqrt{33}\cos\theta$

$\underline{a}\cdot\underline{b} = (3,2,1)\cdot(4,-1,4) = 12-2+4 = 14$

$\Big\}\ 14 = \sqrt{14}\sqrt{33}\cos\theta \Rightarrow \sqrt{\frac{14}{33}} = \cos\theta.$ 1

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(ii) Let area of triangle be denoted by Δ, then $\Delta = \frac{1}{2}|\underline{a}\times\underline{b}|$

where $\underline{a}\times\underline{b} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 2 & 1 \\ 4 & -1 & 4 \end{vmatrix} = \underline{i}\begin{vmatrix} 2 & 1 \\ -1 & 4\end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 1 \\ 4 & 4\end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 2 \\ 4 & -1\end{vmatrix}$

$= \underline{i}(2.4-1.-1) - \underline{j}(3.4-1.4) + \underline{k}(3.-1-2.4)$

$= \underline{i}(8+1) - \underline{j}(12-4) + \underline{k}(-3-8)$

$= 9\underline{i} - 8\underline{j} - 11\underline{k}$

$= (9,-8,-11)$ 2

$\frac{1}{2}|\underline{a}\times\underline{b}| = \frac{1}{2}\sqrt{9^2+(-8)^2+(-11)^2} = \frac{1}{2}\sqrt{81+64+121} = \frac{1}{2}\sqrt{266} = \frac{\sqrt{2}}{2}\sqrt{133}$

$= \sqrt{\frac{133}{2}} = 8.15$ 1

to 2 dp. $\overline{3}$

3. (i) $\underline{c}\cdot\hat{\underline{a}} = (\hat{\underline{b}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})\hat{\underline{a}})\cdot\hat{\underline{a}}$

$= \hat{\underline{b}}\cdot\hat{\underline{a}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})\hat{\underline{a}}\cdot\hat{\underline{a}}$

$= \hat{\underline{b}}\cdot\hat{\underline{a}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})1 = 0$, so are orthogonal. 2

3 (ii) $\underline{c}\cdot\underline{c} = |\underline{c}|^2$ so $\sqrt{\underline{c}\cdot\underline{c}} = |\underline{c}|$,

$$\underline{c}\cdot\underline{c} = (\hat{\underline{b}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})\hat{\underline{a}})\cdot(\hat{\underline{b}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})\hat{\underline{a}})$$
$$= \hat{\underline{b}}\cdot\hat{\underline{b}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})\hat{\underline{a}}\cdot\hat{\underline{b}} - (\hat{\underline{a}}\cdot\hat{\underline{b}})\hat{\underline{b}}\cdot\hat{\underline{a}} + (\hat{\underline{a}}\cdot\hat{\underline{b}})^2\hat{\underline{a}}\cdot\hat{\underline{a}}$$
$$= 1 - 2(\hat{\underline{a}}\cdot\hat{\underline{b}})^2 + (\hat{\underline{a}}\cdot\hat{\underline{b}})^2 = 1-(\hat{\underline{a}}\cdot\hat{\underline{b}})^2$$

As $\hat{\underline{a}}\cdot\hat{\underline{b}} = |\hat{a}|\,|\hat{b}|\cos\theta$, and $|\hat{a}| = |\hat{\underline{b}}| = 1$, then

$$\underline{c}\cdot\underline{c} = 1 - \cos^2\theta = \sin^2\theta = |\underline{c}|^2 \Rightarrow |\underline{c}| = \sin\theta \quad 2$$

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4 (i) Let $A = (1,2,1)$, $B = (3,0,4)$ and let $\underline{a} = (1,2,1)$
$\underline{v} = \vec{AB} = (-2,2,3)$

$$\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) \Rightarrow \begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 3\\ 0\\ 4\end{pmatrix} + \lambda\begin{pmatrix} -2\\ 2\\ -3\end{pmatrix} \quad 1$$

$$(*) \Rightarrow \begin{cases} x = 3-2\lambda \\ y = 2\lambda \\ z = 4-3\lambda\end{cases} \Rightarrow \lambda = \frac{x-3}{-2} = y/2 = \frac{z-4}{-3} \quad 1$$

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(ii) Π: $\underline{r} = \underline{a} + \lambda(\underline{b}-\underline{a}) + \mu(\underline{c}-\underline{a})$

$$\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 1\\ 1\\ 1\end{pmatrix} + \lambda\begin{pmatrix} -1\\ -3\\ -1\end{pmatrix} + \mu\begin{pmatrix} 0\\ -1\\ 3\end{pmatrix} \Rightarrow \begin{cases} x = 1-\lambda & (1)\\ y = 1-3\lambda-\mu & (2)\\ z = 1-\lambda+3\mu & (3)\end{cases} \quad 1$$

From (1) $\lambda = 1-x$ hence substituting into (2) $1-3(1-x)-\mu = y$
$\Rightarrow -2+3x-\mu = y$ 1
Then substitute expressions for λ & μ into (3) $\Rightarrow \mu = -2+3x-y$ — (4)

$$z = 1-(1-x) + 3(-2+3x-y)$$
$$= 1-1+x-6+9x-3y \Rightarrow 10x-3y-z = 6. \quad 1$$

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(iii) Substitute general point on line (*) into equation of plane (1)-(3)

$$10(3-2\lambda) - 3(2\lambda) - 1(4-3\lambda) = 6$$
$$30 - 20\lambda - 6\lambda - 4 + 3\lambda = 6 \Rightarrow -23\lambda = -20 \Rightarrow \lambda = \frac{20}{23}. \quad 1$$

When $\lambda = 20/80$ $\begin{pmatrix} x\\ y\\ z\end{pmatrix} = \begin{pmatrix} 3\\ 0\\ 4\end{pmatrix} + \frac{20}{23}\begin{pmatrix} -2\\ 2\\ -3\end{pmatrix} = (29/23, 40/23, 32/23)$ 1

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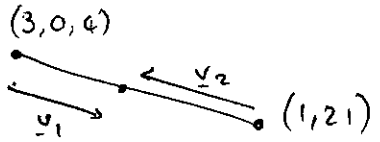
4 (iv) Is $\left(\frac{29}{23}, \frac{40}{23}, 32/23\right)$ nearer $(1,2,1)$ or $(3,0,4)$

The lengths of the vectors between the two points are



$\underline{v}_1 = \frac{1}{23}(69-29, -40, 92-32) = \frac{1}{23}(40, -40, 60)$

$\underline{v}_2 = \frac{1}{23}(29-23, 40-46, 32-23) = \frac{1}{23}(6, -6, 9)$

$$\sqrt{\frac{1}{23^2}\left(40^2 + (-40)^2 + 60^2\right)} > \sqrt{\frac{1}{23^2}\left(6^2 + (-6)^2 + 9^2\right)}$$

So is nearer $(1,2,1)$. 2

5. Volume $V = |\underline{a} \cdot (\underline{b} \times \underline{c})|$ 1

$$\underline{b} \times \underline{c} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 3 & 1 & 0 \\ 1 & 1 & 5 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & 0 \\ 1 & 5 \end{vmatrix} - \underline{j}\begin{vmatrix} 3 & 0 \\ 1 & 5 \end{vmatrix} + \underline{k}\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix}$$

$$= 5\underline{i} - 15\underline{j} + (3-1)\underline{k}$$

$$= 5\underline{i} - 15\underline{j} + 2\underline{k}$$

$$= (5, -15, 2) \quad 2$$

$|a \cdot (b \times c)| = |(2,0,0) \cdot (5,-15,2)| = 10.$ 1

$\overline{4}$

6. (i) normal vectors to the planes are $(2,1,-3)$ and $(\alpha, 1, 1)$

If normals are orthogonal, so will planes be, hence 1

$$(2,1,-3) \cdot (\alpha, 1, 1) = 0$$

$$\Rightarrow 2\alpha + 1 - 3 = 0 = 2\alpha - 2 = 2(\alpha - 1) = 0$$

$$\Rightarrow \alpha = 1. \quad 1$$

$\overline{2}$

(ii) As $\alpha = 1$, then $x + y + z = 3$, Solve $\begin{cases} 2x + y - 3z = 8 \\ x + y + z = 3 \end{cases}$ let $z = 0$ $\begin{cases} 2x + y = 8 \\ x + y = 3 \end{cases} \Rightarrow x = 5, \ y = -2$ 1

A vector in the direction of the line of intersection will be orthogonal to both normal vectors: With a point on the line and its direction we can find the equation of the line.

$$\begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{vmatrix} = \underline{i}\begin{vmatrix} 1 & -3 \\ 1 & 1 \end{vmatrix} - \underline{j}\begin{vmatrix} 2 & -3 \\ 1 & 1 \end{vmatrix} + \underline{k}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = (4, -5, 1), \quad 2$$

hence $\underline{r} = \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ -5 \\ 1 \end{pmatrix}.$

$\overline{3}$

[END]