MECH1010 : Modelling and Analysis in Engineering I
Linear Algebra

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Contents

1 Introduction

The aim of this part of the course is to cover the basics of vectors in two- and three-dimensions. It will also provide a brief introduction to matrices. You will continue to study vectors and matrices next year in 2010: Modelling and Analysis II. You should all have covered vectors in two dimensions before, and have seen them in complex numbers to some extent. The course will start from scratch, but will cover the basics very quickly. The aim of these notes is to refresh your memories of some of the fundamentals and to formally layout some of the basic definitions.

Recommended Reading

- K. A. Stroud, *Engineering Mathematics* London: Palgrave Macmillan, 6^{th} Revised edition.

*This document can be downloaded from: http://www.ucl.ac.uk/~ucesdsi/teaching.html

2 Vectors

A **vector** is a quantity with both direction *and* magnitude. An example of a vector is displacement, which is a distance from a point to another point. Another example is velocity, which is a vector and whereas speed is the associated scalar quantity.

Symbol	Notation
$\boldsymbol{a}$, $\vec{a}$, $\underline{a}$, $\overrightarrow{OA}$	Vector
$\lvert\boldsymbol{a}\rvert$, $\lVert\boldsymbol{a}\rVert$	Magnitude or Norm of a vector
$\hat{\boldsymbol{a}}$	Unit vector
$(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, $(\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z)$	Orthonormal basis vectors

A vector can be defined by an initial point O and an end point A as $\overrightarrow{OA}$. The notion of direction implies a spatial frame of reference, i.e. a displacement from a start point or a velocity in a direction. Thus, vectors can have **dimension** determined by the frame in which we are working. For example the points A and B may lie in a (x, y)-plane as $A = (x_1, y_1)$ and $B = (x_2, y_2)$ so that $\overrightarrow{AB} = (x_2 - x_1, y_2 - y_1)$. Alternatively a vector may exist in three-dimensions or arbitrarily many dimensions. A vector is written as a row or a column of numbers, whose i^{th}-entries are denoted with the i^{th}-subscript.

$$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \quad \text{or} \quad \boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

In general a **position vector** is written horizontally and a **direction vector** vertically. A position vector is fixed with respect to the origin, whereas a direction vector is not fixed.

The magnitude of a vector $\boldsymbol{x}$ is given by the scalar quantity c as

$$|\boldsymbol{x}| = \sqrt{x_1^2 + x_2^2 + \ldots + x_n^2} = \sqrt{\sum_{i=1}^{n} x_i^2} = c \tag{1}$$

The magnitude of a vector can be visualised as its length. The magnitude of the vector $\overrightarrow{AB}$ is equal to the magnitude of the vector $\overrightarrow{BA}$ as the distances are the same, i.e. $|\overrightarrow{AB}| = |\overrightarrow{BA}|$. The directions are different as the initial frames of reference are different: one starts from A and ends at B, the other in the opposite direction.

A vector whose magnitude is equal to one is said to be a **unit vector** and is commonly denoted by $\hat{\boldsymbol{v}}$. Thus any vector can be normalized, that is turned into a unit vector, by

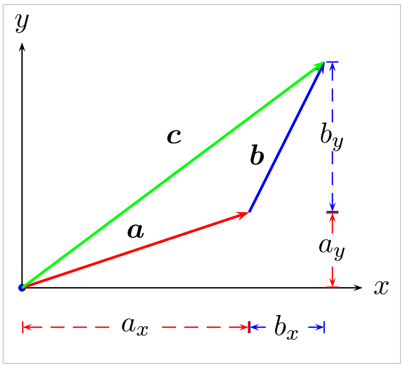
$$\hat{\boldsymbol{v}} = \frac{\boldsymbol{v}}{|\boldsymbol{v}|}. \tag{2}$$

2.1 Addition and Subtraction of Vectors

Vectors of the same dimension can be added and subtracted. For these operations we use the parallelogram law. We may add vectors of the same dimension in the usual way $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{c}$, where the new vector $\boldsymbol{c}$ is defined by its individual elements as

$$c_i = a_i + b_i \quad \text{for} \quad i = 1, 2, \ldots n. \tag{3}$$

This is the same as $\overrightarrow{AB} + \overrightarrow{BC}$ being equal to $\overrightarrow{AC}$. Thus, if we add $\boldsymbol{v}_1 = \overrightarrow{AB}$ to $\boldsymbol{v}_2 = \overrightarrow{BA}$, then $\boldsymbol{v}_1 + \boldsymbol{v}_2 = \boldsymbol{0}$ (the zero vector is the vector whose entries are all zero) as $\boldsymbol{v}_1 = -\boldsymbol{v}_2$.



Vectors have the following general properties:

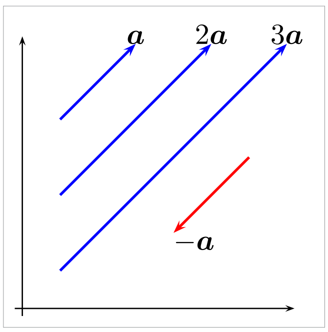
1. Addition of vectors is **commutative**, that is $\boldsymbol{a} + \boldsymbol{b} = \boldsymbol{b} + \boldsymbol{a}$

2. Addition is also **associative**, i.e. $(\boldsymbol{a} + \boldsymbol{b}) + \boldsymbol{c} = \boldsymbol{a} + (\boldsymbol{b} + \boldsymbol{c})$

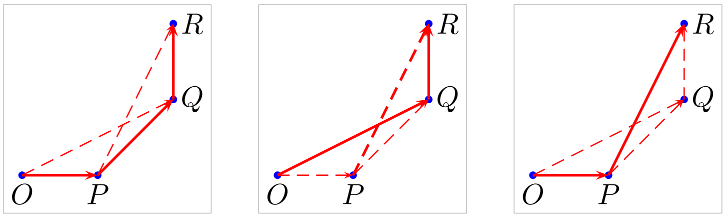
3. Vectors can be multiplied by scalars. This can change both magnitude and direction.

   (i) The sign of the scalar can change the direction of a vector : if the scalar is negative the direction is reversed.

   (ii) The size of the scalar can change the magnitude of a vector : if the scalar is greater than one the magnitude is increased.



To show that addition of vectors is associative consider the system $OPQR$



There are multiple paths from O to R

$$\left(\overrightarrow{OP}+\overrightarrow{PQ}\right)+\overrightarrow{QR}=\overrightarrow{OQ}+\overrightarrow{QR} = \overrightarrow{OR} \qquad \text{and} \qquad \overrightarrow{OP}+\left(\overrightarrow{PQ}+\overrightarrow{QR}\right)=\overrightarrow{OP}+\overrightarrow{PR} = \overrightarrow{OR}$$

2.2 Components of a Vector Or How to Establish a Co-ordinate System

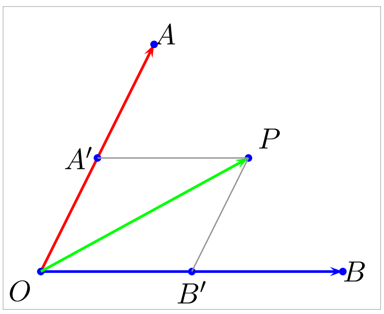
In three-dimensional Cartesian co-ordinates the unit vectors that define the co-ordinate system are denoted by $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ or by $\boldsymbol{e_x}$, $\boldsymbol{e_y}$ and $\boldsymbol{e_z}$. Often vectors expressed in this system are denoted by $\boldsymbol{a}=(a_x,a_y,a_z)=a_x\boldsymbol{e_x}+a_y\boldsymbol{e_y}+a_z\boldsymbol{e_z}=a_1\boldsymbol{i}+a_2\boldsymbol{j}+a_3\boldsymbol{k}$. The components of a vector are determined by the frame chosen.

Two Dimensions

Given three vectors $\boldsymbol{a}=\overrightarrow{OA}$, $\boldsymbol{b}=\overrightarrow{OB}$ and $\boldsymbol{p}=\overrightarrow{OP}$, where $\boldsymbol{p}$ lies in the plane of $\boldsymbol{a}$ and $\boldsymbol{b}$ and hence may be expressed as $\overrightarrow{OP}=\overrightarrow{OA'}+\overrightarrow{OB'}$ for points A' and B' which lie on $\boldsymbol{a}$ and $\boldsymbol{b}$. Let $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ be the unit vectors of $\boldsymbol{a}$ and $\boldsymbol{b}$. Let $p_1=|\overrightarrow{OA'}|$ and $p_2=|\overrightarrow{OB'}|$, hence by rescaling the unit vectors we may express the vector $\boldsymbol{p}$ as

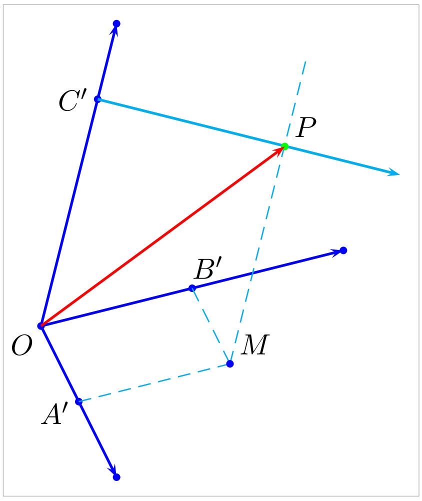
$$\begin{aligned}\overrightarrow{OP}&=\overrightarrow{OA'}+\overrightarrow{OB'}\\&=p_1\hat{\boldsymbol{a}}+p_2\hat{\boldsymbol{b}}.\end{aligned}$$

Then $P=(p_1,p_2)$ are called the co-ordinates of P with respect to the axes $\overrightarrow{OA}$ and $\overrightarrow{OB}$, where $\overrightarrow{OP}$ is written as the sum of its components along the axes $\overrightarrow{OA}$ and $\overrightarrow{OB}$.



Three-Dimensions

Consider three points A, B and C with vectors $\boldsymbol{a} = \overrightarrow{OA}$, $\boldsymbol{b} = \overrightarrow{OB}$ and $\boldsymbol{c} = \overrightarrow{OC}$ which are non-coplanar.



Consider a point P. Then let $\overrightarrow{PM}$ be the line through P which is parallel to $\boldsymbol{c}$. This line intersects the plane formed by $\boldsymbol{a}$ and $\boldsymbol{b}$ at the point M. Let the line which is parallel to $\overrightarrow{OM}$ and passes through P intersect the vector $\boldsymbol{c}$ at point C'. Hence

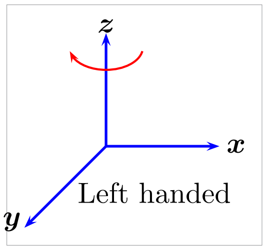
$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OM} + \overrightarrow{OC'} \\ &= \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} \\ &= p_1\hat{\boldsymbol{a}} + p_2\hat{\boldsymbol{b}} + p_3\hat{\boldsymbol{c}}.\end{aligned}$$

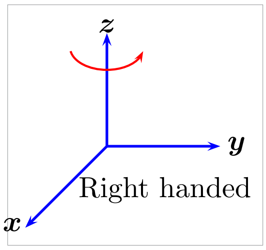
Thus, (p_1, p_2, p_3) are the co-ordinates of P with respect to the $\overrightarrow{OA}$, $\overrightarrow{OB}$ and $\overrightarrow{OC}$ frame, i.e.

$$\boldsymbol{p} = p_1\hat{\boldsymbol{a}} + p_2\hat{\boldsymbol{b}} + p_3\hat{\boldsymbol{c}}.$$

If the vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are mutually perpendicular then the frame will be an orthogonal coordinate system. There are two possible configurations for an orthogonal co-ordinate system: right-handed and left-handed. Axes must be taken in cyclic order, using thumb, first and second finger as an aid. The right-handed is the most common (in fact left-hand systems will not be used in this course). For a right-handed system the direction of positive rotation is given by the right-hand screw rule.

Now let the vectors $\hat{\boldsymbol{a}}$, $\hat{\boldsymbol{b}}$ and $\hat{\boldsymbol{c}}$ be the x-, y- and z- axis in three-dimensional space. Thus the position of a point P with respect to the origin can be represented by the position vector (p_x, p_y, p_z) in Cartesian co-ordinates.

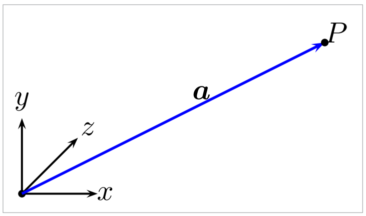




2.3 Applications of Vectors to Elementary Geometry

Point

The point P has position vector $\overrightarrow{OP} = \boldsymbol{a}$ relative to the origin in this example, where $|\boldsymbol{a}|$ is the distance between P and the origin O and the direction is taken from the origin O to P with respect to the local coordinate system.



Line

Suppose a line goes through the point P with position vector $\boldsymbol{a}$ and the line is parallel to the direction vector $\boldsymbol{b}$. A general point Q on the line is given by $\overrightarrow{OQ} = \boldsymbol{r}$ which can be expressed as

$$\boldsymbol{r} = \boldsymbol{a} + \lambda \boldsymbol{b}. \tag{4}$$

In Cartesian coordinates, i.e. the (x, y, z)-frame, let

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \quad \text{and} \quad \boldsymbol{b} = \begin{pmatrix} l \\ m \\ n \end{pmatrix}. \tag{5}$$

Hence by equating components of the vector equation,

$$x = \alpha + \lambda l \quad y = \beta + \lambda m \quad \text{and} \quad z = \gamma + \lambda n.$$

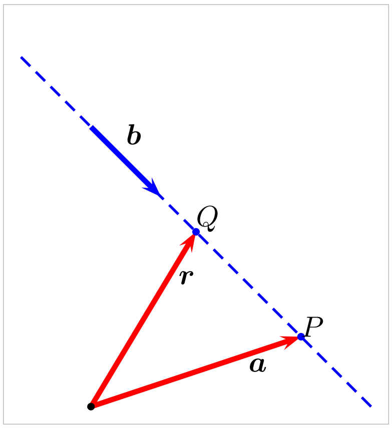
Thus the standard form of the algebraic equation for a line is

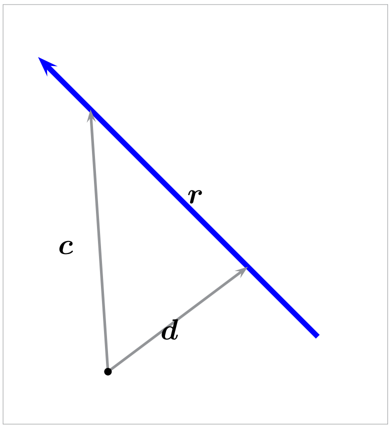
$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = \lambda \tag{6}$$

for a point on the line (α, β, γ) and a vector along the line $(l, m, n)^T$.

Now suppose a line passes through two points which have position vectors $\boldsymbol{c}$ and $\boldsymbol{d}$. The direction of the line connecting the two points may be written as $\boldsymbol{d}-\boldsymbol{c}$. The line passes through the point $\boldsymbol{c}$, so the equation of the line may be written as

$$\boldsymbol{r} = \boldsymbol{c} + \lambda\,(\boldsymbol{d} - \boldsymbol{c})\,. \tag{7}$$





Example 2.1. *Find the vector and Cartesian equation of the line which passes through the point* $(1, 2, 3)$ *and has direction* $(-1, 1, 4)^T$. *Show that the point* $(-1, 4, 11)$ *also lies on this line. The equation for the line is given by*

$$\boldsymbol{r} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} -1 \\ 1 \\ 4 \end{pmatrix}.$$

Thus $x = 1 - \lambda$, $y = 2 + \lambda$ *and* $z = 3 + 4\lambda$. *Hence*

$$\frac{x-1}{-1} = \frac{y-2}{1} = \frac{z-3}{4} = \lambda.$$

The point $(-1, 4, 11)$ *lies on the line when* $\lambda = 2$. □

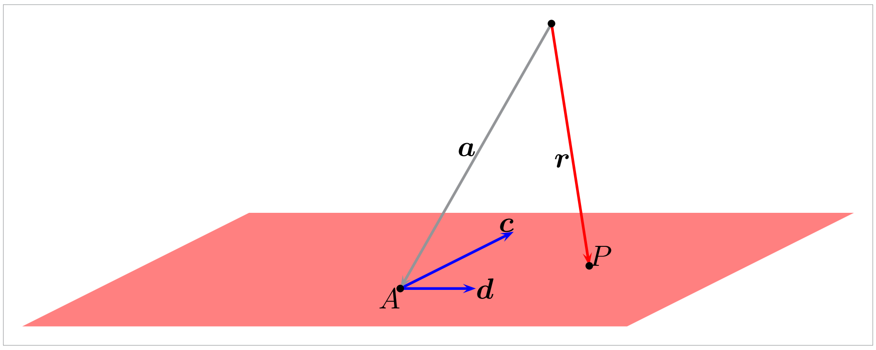
Plane

A plane may be characterised by three non-collinear points A, B and C, with position vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ respectively. The equation of the plane may then be formulated by a vector which defines every point on the plane

$$\boldsymbol{r} = \boldsymbol{a} + \lambda\,(\boldsymbol{b} - \boldsymbol{a}) + \mu\,(\boldsymbol{c} - \boldsymbol{a})\,. \tag{8}$$

Or in algebraic form

$$\Pi\,:\; ax + by + cz = d \tag{9}$$



Example 2.2. *Find the equation of the plane containing the points A, B and C which have position vectors*

$$\boldsymbol{a} = (1, 2, 3)\,, \quad \boldsymbol{b} = (1, -1, 2) \quad \text{and} \quad \boldsymbol{c} = (3, 2, 1)\,.$$

Thus

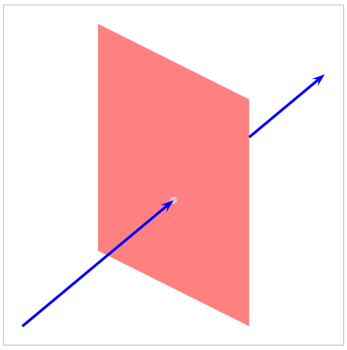
$$\boldsymbol{p} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}.$$

Hence

$$x = 1 - 2\mu \tag{10a}$$
$$y = 2 + 3\lambda \tag{10b}$$
$$z = 3 + \lambda + 2\mu \tag{10c}$$

Now we have three equations with two unknowns. If we solve the first equation (10a) for $\mu = (1 - x)/2$ *and the second equation (10b) for* $\lambda = (y - 2)/3$ *and substitute then both into the third equation (10c) we have* $3x - y + 3z = 10$. □

Example 2.3. *Find the point of intersection between the line given by* $\boldsymbol{r} = (4, 4, 3) + \lambda\,(-1, 1, 4)$ *and the plane given by* $\Pi : 3x - y + 3z = 10$.



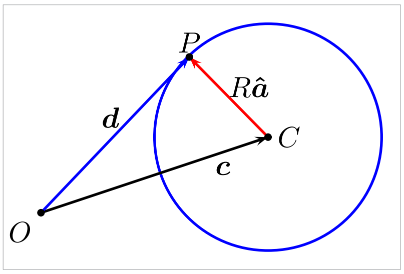
The equation of line can be expressed as $4 - x = y - 4 = (z - 3)/4 = \lambda$. *Hence a point on the line is given by* $x = -\lambda$, $y = \lambda + 4$ *and* $z = 4\lambda + 3$. *Substituting these values into the equation* *for the plane gives*

$$\begin{aligned}10 &= 3\,(4-\lambda) - (4+\lambda) + 3\,(4\lambda+3)\\ &= 12 - 3\lambda - 4 - \lambda + 9 + 12\lambda\\ &= 17 + 8\lambda.\end{aligned}$$

Hence $\lambda = -7/8$ *and the point on the plane is given by* $(39/8, 25/8, -1/2)$. □

Sphere

The equation of a sphere of radius R centred at point C is given by $R\hat{\boldsymbol{a}} = \boldsymbol{d} - \boldsymbol{c}$.



2.4 The Dot Product

The **dot product** is a way of multiplying vectors with a scalar solution, for this reason it is occasionally called the **scalar product**. The dot product is defined as

$$\boldsymbol{a}\cdot\boldsymbol{b} = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \ldots + a_n b_n = c. \tag{11}$$

Hence, we may write the magnitude of a vector as

$$|\boldsymbol{v}| = \sqrt{\boldsymbol{v}\cdot\boldsymbol{v}} \quad \Rightarrow \quad |\boldsymbol{v}|^2 = \boldsymbol{v}\cdot\boldsymbol{v} = \sum_{i=1}^{n} v_i^2. \tag{12}$$

Thus we have that the dot product is **commutative**, that is $\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{b}\cdot\boldsymbol{a}$.

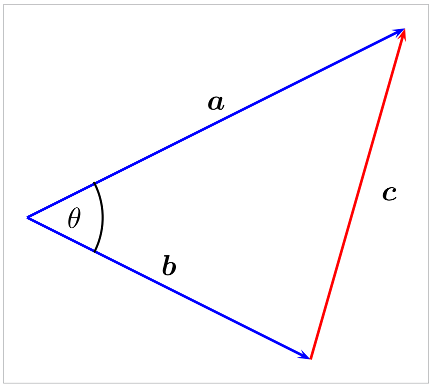
Now consider two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ extending from the origin, separated by an angle θ. A third vector $\boldsymbol{c}$ may be defined as

$$\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}.$$

creating a triangle with sides $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

By the cosine rule

$$\begin{aligned}\boldsymbol{c}\cdot\boldsymbol{c} = |\boldsymbol{c}|^2 &= |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta\\ &= \boldsymbol{a}\cdot\boldsymbol{a} + \boldsymbol{b}\cdot\boldsymbol{b} - 2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta.\end{aligned}$$



But as $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b}$, we also have

$$\begin{aligned}\boldsymbol{c}\cdot\boldsymbol{c} &= (\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})\\ &= \boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}-2\,\boldsymbol{a}\cdot\boldsymbol{b}.\end{aligned}$$

Hence comparing the two expressions

$$\boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}-2\,\boldsymbol{a}\cdot\boldsymbol{b} = \boldsymbol{a}\cdot\boldsymbol{a}+\boldsymbol{b}\cdot\boldsymbol{b}-2|\boldsymbol{a}||\boldsymbol{b}|\cos\theta.$$

Thus, on re-arranging

$$\boldsymbol{a}\cdot\boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta. \tag{13}$$

So the dot product between two vectors can be used to find the angle between them as

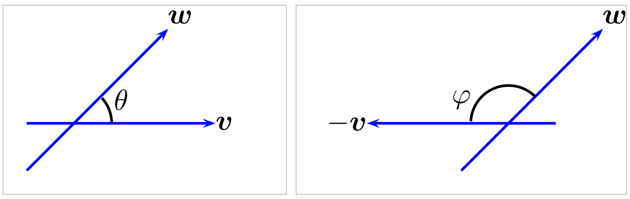
$$\theta = \cos^{-1}\left(\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\right). \tag{14}$$

The angle measured by $\boldsymbol{a}\cdot\boldsymbol{b}$ is measured in the opposite direction to the angle measured by $\boldsymbol{b}\cdot\boldsymbol{a}$ as one is measured from the reference vector defined by $\boldsymbol{a}$ the other $\boldsymbol{b}$ but the two dot products are equal as $\cos(-\theta) = \cos(\theta)$.

Thus for two non-zero vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ if $\boldsymbol{a}\cdot\boldsymbol{b} = 0$ then $\cos\theta = 0$, that is the angle between them is $\pm\pi/2$, i.e. the two vectors are at right angles. Such vectors are said to be **orthogonal**. If both vectors are unit vectors, i.e. $\boldsymbol{a} = \hat{\boldsymbol{a}}$ and $\boldsymbol{b} = \hat{\boldsymbol{b}}$, such vectors are said to **orthonormal**.

Note that the dot product provides the angle between two positive facing vectors.

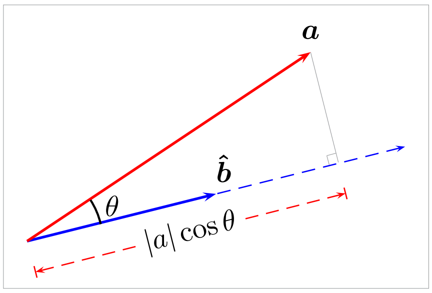
Example 2.4. *What is the angle, θ, between $\boldsymbol{v} = (1, 0)$ and $\boldsymbol{w} = (1, 1)$? What is the angle, φ, between $\boldsymbol{w}$ and $-\boldsymbol{v}$?*



As $|\pm \boldsymbol{v}| = 1$ *and* $|\boldsymbol{w}| = \sqrt{2}$ *and* $\pm\boldsymbol{v}\cdot\boldsymbol{w} = \pm 1$ *then the angles are given by* $\theta = \cos^{-1}\frac{-1}{\sqrt{2}} = \frac{\pi}{4}$ *and* $\varphi = \cos^{-1}\frac{-1}{\sqrt{4}} = \frac{3\pi}{4}$. □

For two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ the **vector projection** of $\boldsymbol{a}$ onto $\boldsymbol{b}$ is given by $\left(\boldsymbol{a}\cdot\hat{\boldsymbol{b}}\right)\hat{\boldsymbol{b}} = (|\boldsymbol{a}|\cos\theta)\hat{\boldsymbol{b}}$.

The dot product $\boldsymbol{a}\cdot\hat{\boldsymbol{b}} = |\boldsymbol{a}|\cos\theta$, i.e., the magnitude of the projection of $\boldsymbol{a}$ in the direction of $\hat{\boldsymbol{b}}$. This is called the **scalar projection** of $\boldsymbol{a}$ onto $\hat{\boldsymbol{b}}$, or scalar component of $\boldsymbol{a}$ in the direction of $\hat{\boldsymbol{b}}$.



Let $\boldsymbol{p}$ be a vector in the Cartesian frame, so that $\boldsymbol{i} = (1,0,0)$, $\boldsymbol{j} = (0,1,0)$ and $\boldsymbol{k} = (0,0,1)$, then

$$\boldsymbol{p} = p_1\boldsymbol{i} + p_2\boldsymbol{j} + p_3\boldsymbol{k}.$$

Let θ_1 the angle between $\boldsymbol{p}$ and $\boldsymbol{i}$, then

$$\begin{aligned}\cos\theta_1 &= \frac{\boldsymbol{i}\cdot\boldsymbol{p}}{|\boldsymbol{i}||\boldsymbol{p}|}\\ &= \frac{p_1}{|\boldsymbol{p}|}.\end{aligned}$$

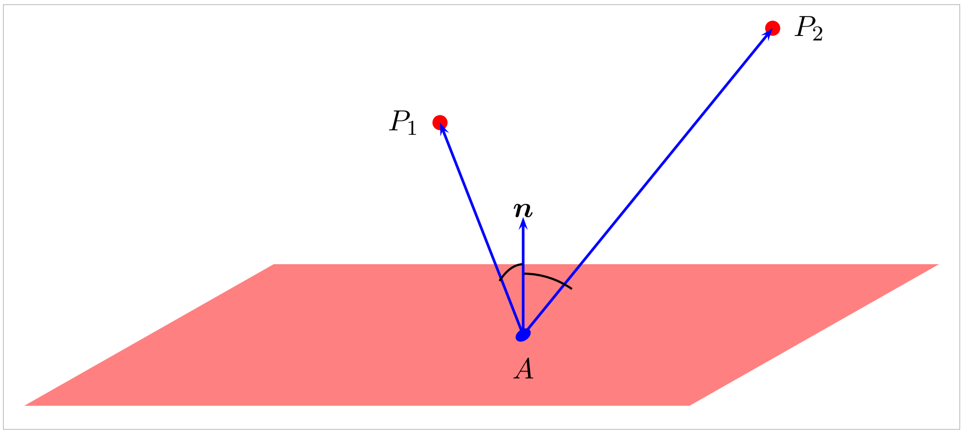
Similarly, the angle θ_2 between $\boldsymbol{p}$ and $\boldsymbol{j}$ is given by $\cos\theta_2 = p_2/|\boldsymbol{p}|$ and the angle θ_3 between $\boldsymbol{p}$ and $\boldsymbol{k}$ is given by $\cos\theta_3 = p_3/|\boldsymbol{p}|$. These are called the **direction cosines** of the vector $\boldsymbol{p}$ with respect to the axes $(\boldsymbol{i},\boldsymbol{j},\boldsymbol{k})$. Note that $\cos^2\theta_1 + \cos^2\theta_2 + \cos^2\theta_3 = 1$.

Example 2.5. *Are the points* $P_1 = (1,3,-2)$ *and* $P_2 = (4,-2,1)$ *on the same side of the plane* $3x + 4y - z = 1$*?*

Let A *be a point on the plane and* $\boldsymbol{n}$ *be a vector normal to the plane. Let* $\boldsymbol{x} = \overrightarrow{AP_1}$ *and* $\boldsymbol{y} = \overrightarrow{AP_2}$*. Consider* $\boldsymbol{x}\cdot\boldsymbol{n}$ *and* $\boldsymbol{y}\cdot\boldsymbol{n}$*. If both points are on the same side of the plane, the sign of the scalar products will be the same. Let* $A = (0,0,-1)$ *be a point on the plane, then* $\boldsymbol{n} = (3,4,-1)$ *and* $\boldsymbol{x} = (1,3,-2)$ *and* $\boldsymbol{y} = (4,2,-2)$*. So*

$$\boldsymbol{x}\cdot\boldsymbol{n} = (1,3,-1)\cdot(3,4,-1) = 16 \quad \textit{and} \quad \boldsymbol{y}\cdot\boldsymbol{n} = (4,2,-2)\cdot(3,4,-1) = 22.$$

So both points are on the same side of the plane. □

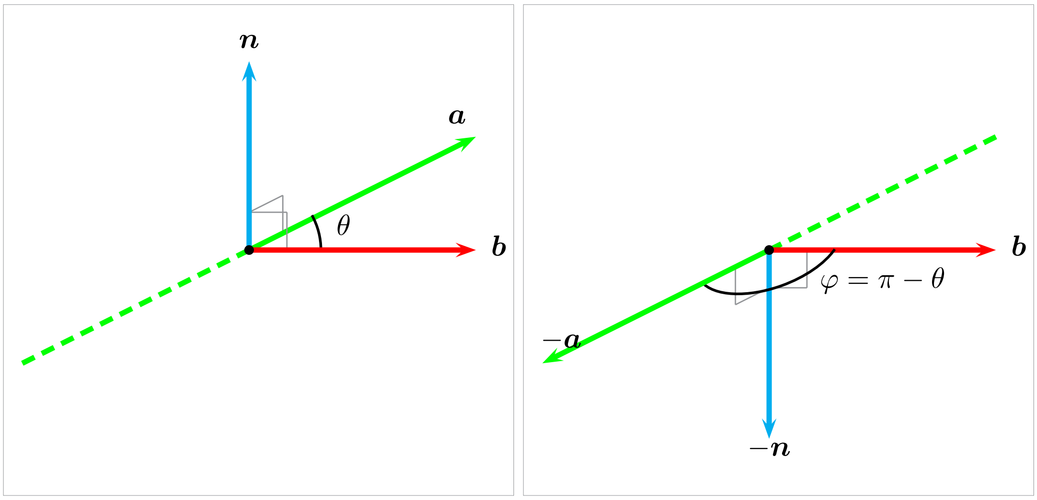


2.5 The Cross Product

As we have seen, any two non-parallel vectors, $\boldsymbol{a}$, $\boldsymbol{b}$ can be used to define a plane. A unit vector $\hat{\boldsymbol{n}}$ which is normal to the plane can be found by the **cross product** (often called **vector product**, as the output of this operation is a vector) as

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{n} \\ &= \hat{\boldsymbol{n}} |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \end{aligned} \tag{15}$$

where θ is the smaller angle between the two vectors, that is $0 \leq \theta \leq \pi$.



For a pair of non-parallel vectors $\boldsymbol{a} = (a_1, a_2, a_3)$ and $\boldsymbol{b} = (b_1, b_2, b_3)$, in a Cartesian frame $(\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ then the vector product is given by $\boldsymbol{n} = (n_1, n_2, n_3) = n_1\boldsymbol{i} + n_1\boldsymbol{j} + n_3\boldsymbol{k}$ as

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= \boldsymbol{n} \\ &= \hat{\boldsymbol{n}} |\boldsymbol{a}||\boldsymbol{b}| \sin\theta \\ &= (a_2b_3 - a_3b_2)\,\boldsymbol{i} - (a_1b_3 - a_3b_1)\,\boldsymbol{j} + (a_1b_2 - a_2b_1)\,\boldsymbol{k}. \end{aligned} \tag{16}$$

The magnitude of the vector $\boldsymbol{n}$ given by the cross product is equal to the area of the parallelogram with sides $\boldsymbol{a}$ and $\boldsymbol{b}$. However the order in which the cross product is taken is significant as the cross product is **anti-commutative**, that is $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$.

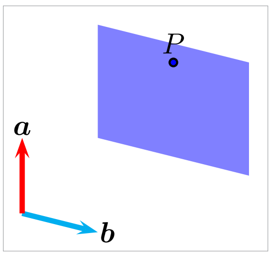
If $|\boldsymbol{a}| \neq 0$ and $|\boldsymbol{b}| \neq 0$ but $\boldsymbol{a} \times \boldsymbol{b} = \boldsymbol{0}$ then $\sin\theta = 0$, so either $\theta = 0$ or $\theta = \pi$, i.e. the vectors are parallel or anti-parallel.

Example 2.6. *Find the area of the triangle with vertices* $A = (1, 1, 1)$*,* $B = (2, 3, -2)$ *and* $C = (1, 2, 4)$*.*
Let $\boldsymbol{r} = \overrightarrow{AB} = (1, 2, -3)$ *and* $\boldsymbol{s} = \overrightarrow{AC} = (0, 1, 3)$*. The area of the triangle is given by* $\Delta = \frac{1}{2}|\boldsymbol{r}||\boldsymbol{s}|\sin\theta = \frac{1}{2}|\boldsymbol{r} \times \boldsymbol{s}|$*.*

$$\begin{aligned} \boldsymbol{r} \times \boldsymbol{s} &= (1, 2, -3) \times (0, 1, 3) \\ &= (2 \star 3 - (-3) \star 1, -1 \star 3 - (-3) \star 0, 1 \star 1 - 2 \star 0) \\ &= (9, -3, 1) \end{aligned}$$

Thus area of the triangle is $\Delta = \frac{1}{2}\sqrt{9^2 + 3^2 + 1^2} = \frac{1}{2}\sqrt{81 + 9 + 1} = \frac{\sqrt{91}}{2}$. □

Example 2.7. *Write down the equation of the plane through the point* $(1, 2, 3)$ *and parallel to the vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$*, where* $\boldsymbol{a} = (1, 0, -1)^T$ *and* $\boldsymbol{b} = (3, 1, 1)^T$*.*



As

$$\boldsymbol{a} \times \boldsymbol{b} = (1, 0, -1)^T \times (3, 1, 1)^T = (1, -4, 1)^T$$

then the general equation of a plane is then given by $x - 4y + z = d$*. On substituting* $x = 1$*,* $y = 2$ *and* $z = 3$*, then* $d = -4$ *so that the equation for the plane becomes* $x - 4y + z = -4$*.*

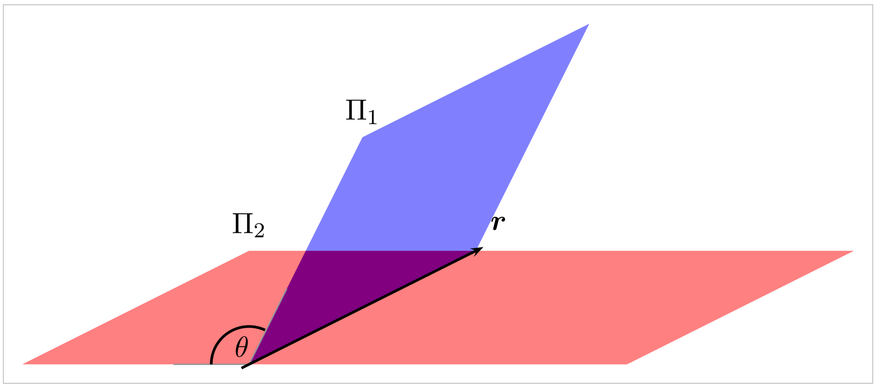
Example 2.8. *Find the angle between the two planes* Π_1 *and* Π_2 *and find the line of intersection, when*

$$\Pi_1 \,:\, y + 3z = 0 \quad \textit{and} \quad \Pi_2 \,:\, x + 2y - 3z = 4.$$

Let the angle between the two planes Π_1 *and* Π_2 *be* φ*, which is given by* $\varphi = \pi - \theta$*, where* θ *is the angle between their two normals,* $\boldsymbol{n_1}$ *and* $\boldsymbol{n_2}$*, where* $\boldsymbol{n_1} = (0, 1, 3)^T$ *and* $\boldsymbol{n_2} = (1, 2, -3)^T$*.*

Thus

$$
\begin{aligned}
\theta &= \cos^{-1}\left(\frac{\boldsymbol{n_1}\cdot\boldsymbol{n_2}}{|\boldsymbol{n_1}||\boldsymbol{n_2}|}\right)\\
&= \cos^{-1}\left(\frac{0+2-9}{\sqrt{10}\sqrt{14}}\right)\\
&= \cos^{-1}\left(\frac{-\sqrt{7}}{2\sqrt{5}}\right)\\
\Rightarrow \varphi &= \pi - \cos^{-1}\left(\frac{-\sqrt{7}}{2\sqrt{5}}\right).
\end{aligned}
$$



The vector along the line of intersection is given by $\boldsymbol{n_1}\times\boldsymbol{n_2} = (-9,3,1)^T$. *A point on the line of intersection can be found by solving the under-determined pair of simultaneous equations for the planes. For the equation for* Π_1 *let* $y=3$, *hence* $z=-1$. *Then substituting these values into* Π_2 *gives* $x=-5$. *Thus the line of intersection is given by*

$$
\boldsymbol{r} = \begin{pmatrix}-5\\3\\-1\end{pmatrix} + \lambda\begin{pmatrix}-9\\3\\1\end{pmatrix}. \qquad \square
$$

2.6 Scalar Triple Product

The **scalar triple product** is given by the dot product of one vector with the cross product of two other vectors. The output of the operation is a scalar.

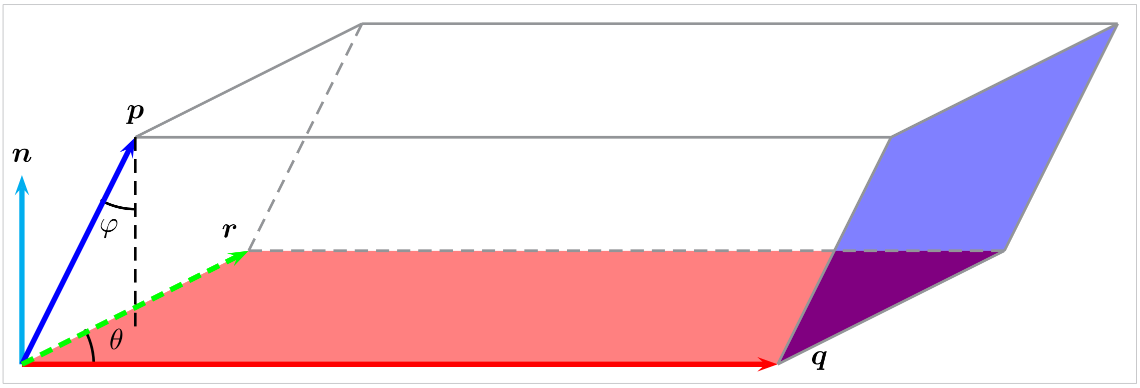
$$
\begin{aligned}
\boldsymbol{p}\cdot(\boldsymbol{q}\times\boldsymbol{r}) &= (\boldsymbol{p}\times\boldsymbol{q})\cdot\boldsymbol{r} &(17)\\
&= (\boldsymbol{r}\times\boldsymbol{p})\cdot\boldsymbol{q} &(18)\\
&= -(\boldsymbol{p}\times\boldsymbol{r})\cdot\boldsymbol{q}. &(19)
\end{aligned}
$$

Strictly speaking the brackets are not necessary as the cross product must be performed first in order to produce another vector on which to apply the dot product.

On remembering that $(\boldsymbol{p}\times\boldsymbol{q}) = -(\boldsymbol{q}\times\boldsymbol{p})$ we note that swapping two of the vectors once simply changes the sign. Hence a cyclic rotation of vectors an even number of times does not alter the sign. For a proof of this property see the appendix.

Geometric Interpretation

As seen $\boldsymbol{q} \times \boldsymbol{r} = \boldsymbol{n} = \hat{\boldsymbol{n}}|\boldsymbol{q}||\boldsymbol{r}|\sin\theta$ is a normal vector whose magnitude is equal to the area of a parallelogram with sides $\boldsymbol{q}$ and $\boldsymbol{r}$ and angle θ.



The volume of the parallelepiped formed by $\boldsymbol{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$ and be calculated by the area of the base parallelogram times the height, which is given by $|\boldsymbol{p}|\cos\varphi$

$$\begin{aligned} \boldsymbol{p} \cdot (\boldsymbol{q} \times \boldsymbol{r}) &= \boldsymbol{n} \cdot \boldsymbol{p} \\ &= |\boldsymbol{n}||\boldsymbol{p}|\cos\varphi \\ &= |\boldsymbol{q}||\boldsymbol{r}||\boldsymbol{p}|\sin\theta\cos\varphi. \end{aligned} \tag{20}$$

2.7 Vector Triple Product

The **vector triple product** is the cross product of one vector with the cross product of another two. The output of the operation is a vector formed as

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \boldsymbol{v} \\ &= (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\,\boldsymbol{c}. \end{aligned} \tag{21}$$

The order and position of the brackets is important

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{c} \times \boldsymbol{b}) &= -\boldsymbol{v}, \\ (\boldsymbol{b} \times \boldsymbol{c}) \times \boldsymbol{a} &= -\boldsymbol{v}, \\ (\boldsymbol{c} \times \boldsymbol{b}) \times \boldsymbol{a} &= \quad \boldsymbol{v} \end{aligned} \tag{22}$$

As $\boldsymbol{v}$ is in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$ then $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ will give a different result to $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})$

$$\begin{aligned} (\boldsymbol{a} \times \boldsymbol{c}) \times \boldsymbol{b} &= (\boldsymbol{b} \cdot \boldsymbol{a})\,\boldsymbol{c} - (\boldsymbol{a} \cdot \boldsymbol{c})\,\boldsymbol{b}, \\ (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} &= (\boldsymbol{c} \cdot \boldsymbol{a})\,\boldsymbol{b} - (\boldsymbol{c} \cdot \boldsymbol{b})\,\boldsymbol{a}. \end{aligned} \tag{23}$$

For a proof of the vector triple product see the appendix.

2.8 Pairs of Lines in Three-Dimensional Spaces

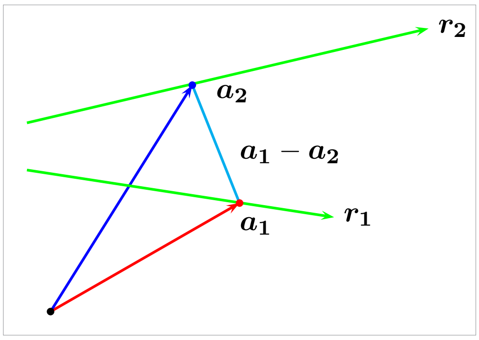
Consider two lines $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ in three-dimensional space given by

$$\boldsymbol{r}_1 \, : \, \boldsymbol{a}_1 + \lambda \boldsymbol{v}_1 \quad \text{and} \quad \boldsymbol{r}_2 \, : \, \boldsymbol{a}_2 + \mu \boldsymbol{v}_2.$$

There are four possible situations:

1. Coincident lines : lines intersect everywhere
2. Intersect at a point : lines intersect once
3. Parallel : lines never intersect
4. Skew : lines never intersect

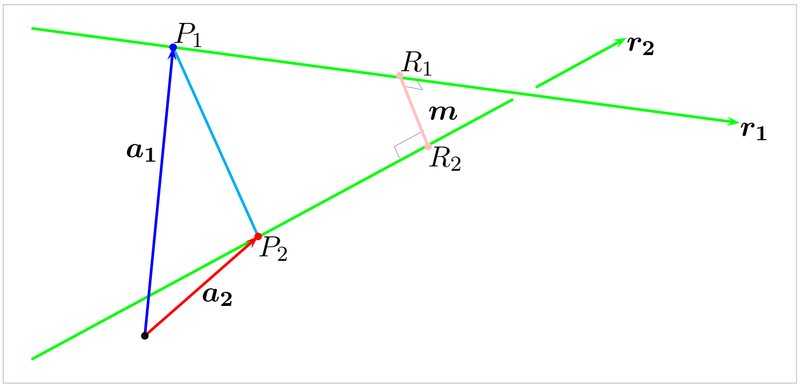
In order to ascertain the number of intersections between a pair of lines there are a number of tests which can be performed systematically:



- Let $|\boldsymbol{v}_1 \times \boldsymbol{v}_2| = 0$ then the direction vectors are parallel, that is $\hat{\boldsymbol{v}}_1 = \pm\hat{\boldsymbol{v}}_2$
  - If $|\,(\boldsymbol{a}_1 - \boldsymbol{a}_2) \times \boldsymbol{v}_1| = 0$ the lines share a common point, hence they are co-incident.
  - Else, if $|\,(\boldsymbol{a}_1 - \boldsymbol{a}_2) \times \boldsymbol{v}_1| \neq 0$ the lines are parallel.
- If $|\boldsymbol{v}_1 \times \boldsymbol{v}_2| \neq 0$ the lines are neither parallel nor coincident. We will test whether they are coplanar.
  - If $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ are coplanar then a normal vector to the plane exists, given by $\boldsymbol{n} = \boldsymbol{v}_1 \times \boldsymbol{v}_2$. Then as $\boldsymbol{a}_1 - \boldsymbol{a}_2$ will lie in the plane, it must be normal to $\boldsymbol{n}$. Hence the dot product between the two vectors must be zero. Thus if $(\boldsymbol{a}_1 - \boldsymbol{a}_2)\cdot(\boldsymbol{v}_1 \times \boldsymbol{v}_2) = 0$ the lines are coplanar and non-parallel, hence they must intersect once.
  - If $(\boldsymbol{a}_1 - \boldsymbol{a}_2)\cdot(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \neq 0$ they are skew and do not intersect at all.

The most common situation is skew lines. In many situations it is of interest to know the shortest distance between a pair of lines and the corresponding points on each line.

The shortest distance between a pair of skew lines is a line mutually perpendicular to them both.



Let $\boldsymbol{m} = \boldsymbol{v}_1 \times \boldsymbol{v}_2$ and consider

$$\begin{aligned}(\boldsymbol{a}_1 - \boldsymbol{a}_2) \cdot \hat{\boldsymbol{m}} &= \overrightarrow{P_2P_1} \cdot \hat{\boldsymbol{m}} \\ &= \left(\overrightarrow{P_2R_2} + \overrightarrow{R_2R_1} + \overrightarrow{R_1P_1}\right) \cdot \hat{\boldsymbol{m}} \\ &= \overrightarrow{R_2R_1} \cdot \hat{\boldsymbol{m}} \\ &= |\overrightarrow{R_2R_1}|.\end{aligned}$$

As $\overrightarrow{P_2R_2}$ is perpendicular to $\hat{\boldsymbol{m}}$ then $\overrightarrow{R_2P_2} \cdot \hat{\boldsymbol{m}} = 0$. Also as $\overrightarrow{R_2R_1}$ is parallel to $\hat{\boldsymbol{m}}$ then $\overrightarrow{R_2R_1} \cdot \hat{\boldsymbol{m}} = |\overrightarrow{R_2R_1}|$. Therefore the length of the common perpendicular is $(\boldsymbol{a}_1 - \boldsymbol{a}_2) \cdot \hat{\boldsymbol{m}}$ which is given by

$$\frac{(\boldsymbol{a}_1 - \boldsymbol{a}_2) \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2)}{|\boldsymbol{v}_1 \times \boldsymbol{v}_2|}. \tag{24}$$

Example 2.9. *Find the minimum distance between the two cylinders A and B, where cylinder A has radius 5 and axis direction $(1, 2, 4)$ which passes through the origin. Cylinder B has radius 3 and axis direction $(2, 1, -3)$ and passes through $(10, -5, 7)$.*

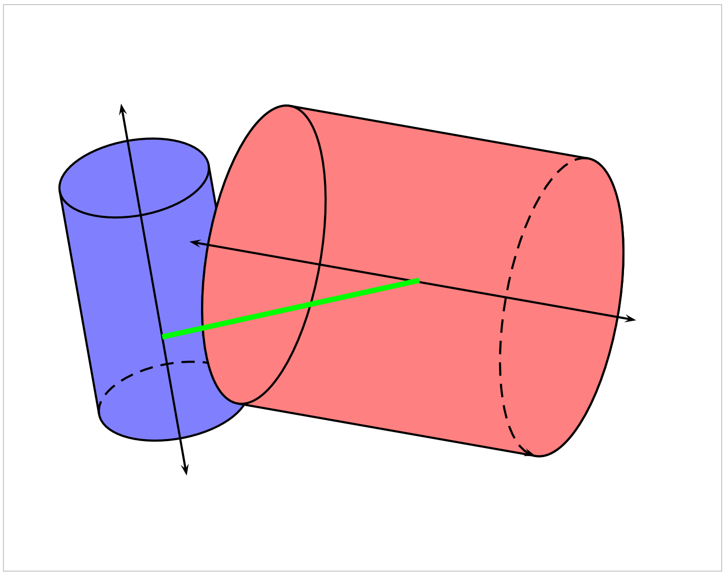
Let $\boldsymbol{v}_1 = (1, 2, 4)^T$ and $\boldsymbol{v}_2 = (2, 1, -3)^T$ with $\boldsymbol{a}_1 = (0, 0, 0)$ and $\boldsymbol{a}_2 = (10, -5, 7)$. Now $\boldsymbol{v}_1 \times \boldsymbol{v}_2 = (-10, 11, -3)^T$, so $|\boldsymbol{v}_1 \times \boldsymbol{v}_2| = \sqrt{10^2 + 11^2 + 3^2} = \sqrt{100 + 121 + 9} = \sqrt{230}$. As $(\boldsymbol{a}_1 - \boldsymbol{a}_2) \cdot (\boldsymbol{v}_1 \times \boldsymbol{v}_2) = (-10, 5, -7)^T \cdot (-10, 11, -3)^T = 100 + 55 + 21 = 176$ then the minimum distance will be $176/\sqrt{230} - 8$. □

Example 2.10. *Find the distance between the two lines*

$$\boldsymbol{r}_1 : x = y - 1 = 4 - z,$$
$$\boldsymbol{r}_2 : x - 2 = \frac{2y + 4}{2} = -z.$$

In the standard form $\dfrac{x - \alpha}{l} = \dfrac{y - \beta}{m} = \dfrac{z - \gamma}{n}$, where (α, β, γ) is a point on the line and $(l, m, n)^T$ a vector along it

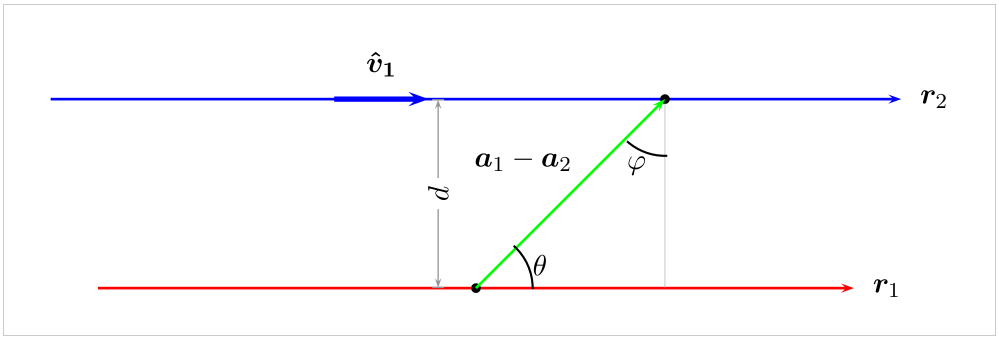
$$\boldsymbol{r}_1 : \frac{x - 0}{1} = \frac{y - 1}{1} = \frac{z - 4}{-1} = \lambda,$$
$$\boldsymbol{r}_2 : \frac{x - 2}{1} = \frac{y + 2}{1} = \frac{z - 0}{-1} = \mu.$$



Thus for both lines $\boldsymbol{v}_1 = \boldsymbol{v}_2 = (1, 1, -1)^T$. *Thus they are parallel. As* $\boldsymbol{a}_1 = (0, 1, 4)$ *and* $\boldsymbol{a}_2 = (2, -2, 0)$ *then* $\boldsymbol{a}_1 - \boldsymbol{a}_2 = (-2, 3, 4)$. *The distance is given by*

$$\begin{aligned} D &= |\boldsymbol{a}_1 - \boldsymbol{a}_2| \cos\theta \\ &= |\boldsymbol{a}_1 - \boldsymbol{a}_2| \sin(\pi/2 - \theta) \\ &= |\hat{\boldsymbol{v}}_1 \times (\boldsymbol{a}_1 - \boldsymbol{a}_2)| \\ &= |\frac{1}{\sqrt{3}}(7, -2, 5)| \\ &= \sqrt{\frac{49 + 4 + 25}{3}} \\ &= \sqrt{26}. \end{aligned}$$

□



3 Matrices

As we have seen vectors can be used to describe a distance and a direction in space. A way of handling vectors, that is scaling them and rotating them, is to use matrices.

This is the geometric definition of a matrix. An alternative application for matrices is to express systems of simultaneous linear equations in a compact, easy to manipulate, form.

A matrix is a rectangular array of numbers. An $(n \times m)$ matrix will have n rows and m columns. The entries of a matrix are called elements. A matrix $\boldsymbol{A}$ will have elements $a_{i,j}$ which denotes the elements on i^{th}-row and j^{th}-column. A matrix is said to be a **square matrix** if $n = m$, i.e. the number of rows is the same as the number of columns.

3.1 Addition of Matrices

Two matrices of the same size can be added together by adding together each individual element. Thus if $\boldsymbol{A}$ and $\boldsymbol{B}$ are $(n \times m)$-matrices then the $(n \times m)$-matrix $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$ has elements $c_{i,j} = a_{i,j} + b_{i,j}$.

3.2 The Transpose of a Matrix

The **transpose** of an $(n \times m)$-matrix $\boldsymbol{A}$ is denoted by $\boldsymbol{A}^T$ and is an $(m \times n)$-matrix comprising of the elements $a_{j,i}$. An matrix which is equal to its own transpose i.e. $\boldsymbol{A} = \boldsymbol{A}^T$ is said to be a **symmetric matrix**. A symmetric matrix must be a square matrix where element-wise $a_{i,j} = a_{j,i}$. A matrix which is equal to $\boldsymbol{A} = -\boldsymbol{A}^T$ is said to be **skew-symmetric matrix** or anti-symmetric matrix. In this case $a_{i,j} = -a_{j,i}$.

3.3 Matrix Multiplication

An $(n \times m)$-matrix $\boldsymbol{A}$ can be multiplied by a scalar α by simply multiplying each element $a_{i,j}$ by α. Thus if $\boldsymbol{B} = \alpha\boldsymbol{A}$ then $b_{i,j} = \alpha a_{i,j}$.

Example 3.1. *A matrix* $\boldsymbol{A}$ *multiplied by* $\alpha = 2$

$$\boldsymbol{A} = \begin{pmatrix} 14 & 2 \\ -1 & -4 \end{pmatrix}, \quad \textit{then} \quad \alpha\boldsymbol{A} = \begin{pmatrix} 28 & 4 \\ -2 & -8 \end{pmatrix} \qquad \square$$

An $(n \times m)$-matrix $\boldsymbol{A}$ can multiply a vector $\boldsymbol{v}$ of length n to give a new vector $\boldsymbol{x}$ of length n, i.e. $\boldsymbol{Ax} = \boldsymbol{x}$ whose elements are given by

$$x_i = a_{i,1}v_i + a_{i,2}v_i + \cdots + a_{i,n}v_i = \sum_{k=1}^{n} a_{i,k}v_i \quad \text{where} \quad 1 \le i \le m.$$

Example 3.2. *For*

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 3 \\ -1 & 5 & \sqrt{2} \end{pmatrix} \quad \textit{and} \quad \boldsymbol{v} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

the matrix-vector product is given by

$$\boldsymbol{x} = \boldsymbol{Av} = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 0 & 3 \\ -1 & 5 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 2+4+10 \\ 15 \\ -2+5+5\sqrt{2} \end{pmatrix} = \begin{pmatrix} 16 \\ 15 \\ 5\sqrt{2}-3 \end{pmatrix}. \qquad \square$$

Matrix-vector multiplication is a specific case of a more general matrix-matrix multiplication on assuming a n-dimensional vector to be a $(n \times 1)$-matrix. However multiplication of two matrices is only well defined when the number of columns of the left matrix is equal to the number of rows of the right matrix. If $\boldsymbol{A}$ is an $(m \times n)$-matrix and $\boldsymbol{B}$ is an $(n \times p)$-matrix, then their matrix product $\boldsymbol{C} = \boldsymbol{AB}$ is the $(m \times p)$-matrix whose entries $c_{i,j}$ are given by the dot-product of the corresponding i^{th}-row of $\boldsymbol{A}$ and the corresponding j^{th}-column of $\boldsymbol{B}$:

$$c_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j} \quad \text{where} \quad 1 \le i \le m \quad \text{and} \quad 1 \le j \le p.$$

A (2×2)-matrix $\boldsymbol{A}$ multiplies a (2×2)-matrix $\boldsymbol{B}$ where

$$\boldsymbol{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{and} \quad \boldsymbol{B} = \begin{pmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{pmatrix}$$

to give a (2×2)-matrix $\boldsymbol{C}$

$$\boldsymbol{C} = \boldsymbol{AB} = \begin{pmatrix} c_{1,1} & c_{1,2} \\ c_{2,1} & c_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1}b_{1,1} + a_{1,2}b_{2,1} & a_{1,1}b_{2,1} + a_{1,2}b_{2,2} \\ a_{2,1}b_{1,1} + a_{2,2}b_{2,1} & a_{2,1}b_{2,1} + a_{2,2}b_{2,2} \end{pmatrix}.$$

If $\boldsymbol{D} = \boldsymbol{BA}$ then the (2×2)-matrix $\boldsymbol{D}$ is given by

$$\boldsymbol{D} = \boldsymbol{BA} = \begin{pmatrix} d_{1,1} & d_{1,2} \\ d_{2,1} & d_{2,2} \end{pmatrix} = \begin{pmatrix} a_{1,1}b_{1,1} + a_{2,1}b_{1,2} & a_{2,1}b_{1,1} + a_{2,2}b_{1,2} \\ a_{1,1}b_{2,1} + a_{2,1}b_{2,2} & a_{2,1}b_{2,1} + a_{2,2}b_{2,2} \end{pmatrix}.$$

A fundamental difference between matrices and integers, real and complex numbers is that matrix multiplication is not commutative, that is $\boldsymbol{AB} \neq \boldsymbol{BA}$.

Example 3.3. *For*

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{B} = \begin{pmatrix} 2 & -1 \\ 1 & \sqrt{2} \end{pmatrix}$$

find $\boldsymbol{AB}$ *and* $\boldsymbol{BA}$.
On multiplication

$$\begin{aligned} \boldsymbol{AB} &= \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & \sqrt{2} \end{pmatrix} = \begin{pmatrix} 2+4 & -1+4\sqrt{2} \\ -1 & -\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 6 & -1+4\sqrt{2} \\ -1 & -\sqrt{2} \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} \boldsymbol{BA} &= \begin{pmatrix} 2 & -1 \\ 1 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 8+1 \\ 1 & 4-\sqrt{2} \end{pmatrix} \\ &= \begin{pmatrix} 2 & 9 \\ 1 & 4-\sqrt{2} \end{pmatrix}. \end{aligned}$$

□

3.4 Types of Matrices

1. A **null matrix**, denoted as $\mathbf{0}$ is an $(n \times m)$-matrix whose elements are all zero, that is $a_{i,j} = 0$ for all $1 \leq i \leq n$, $1 \leq j \leq m$.

2. The **identity matrix** $\mathbb{I}_n$ is an $(n \times n)$-matrix with elements $a_{i,j} = 0$ for all $i \neq j$ and $a_{i,j} = 1$ for all $i = j$.

3. The identity matrix is a specific case of a **diagonal matrix**, which is an $(n \times n)$-matrix with elements $a_{i,j} = 0$ for all $i \neq j$.

4. A **triangular matrix** is a matrix whose elements either above or below the main diagonal are all zero. An **lower triangular matrix** is denoted by $\boldsymbol{L}$ and a **upper triangular matrix** denoted by $\boldsymbol{U}$.

$$\boldsymbol{L} = \begin{pmatrix} l_{1,1} & & & & 0 \\ l_{2,1} & l_{2,2} & & & \\ l_{3,1} & l_{3,2} & \ddots & & \\ \vdots & \vdots & \ddots & \ddots & \\ l_{n,1} & l_{n,2} & \dots & l_{n,n-1} & l_{n,n} \end{pmatrix} \quad \text{and} \quad \boldsymbol{U} = \begin{pmatrix} u_{1,1} & u_{1,2} & u_{1,3} & \dots & u_{1,n} \\ & u_{2,2} & u_{2,3} & \dots & u_{2,n} \\ & & \ddots & \ddots & \vdots \\ & & & \ddots & u_{n-1,n} \\ 0 & & & & u_{n,n} \end{pmatrix}.$$

5. A **rotation matrix** $\boldsymbol{R}(\theta)$ is a $(n \times n)$-matrix which on multiplying a vector rotates the vector by an angle θ. A rotation matrix can be written in the form

$$\boldsymbol{R}(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

6. The **inverse** of an $(n \times n)$-matrix $\boldsymbol{A}$ is an $(n \times n)$-matrix $\boldsymbol{B} = \boldsymbol{A}^{-1}$ such that $\boldsymbol{AB} = \boldsymbol{BA} = \mathbb{I}_n$.

7. A matrix is said to be **singular** if it does not have an inverse.

The inverse of identity matrix is simply the identity matrix. The inverse of a diagonal matrix $\boldsymbol{A}$ whose non-zero elements are $a_{i,i} = \lambda_i$ is the diagonal matrix $\boldsymbol{B}$ whose non-zero elements are given by $b_{i,i} = 1/\lambda_i$. For a general (2×2)-matrix $\boldsymbol{A}$ its inverse is given as

$$\mathbf{A} = \begin{pmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{pmatrix} \quad \text{then} \quad \mathbf{A}^{-1} = \frac{1}{a_{1,1}a_{2,2} - a_{1,2}a_{2,1}} \begin{pmatrix} a_{2,2} & -a_{1,2} \\ -a_{2,1} & a_{1,1} \end{pmatrix}. \tag{25}$$

Thus a (2×2)-matrix is singular when $a_{1,1}a_{2,2} - a_{1,2}a_{2,1} = 0$.

Example 3.4. *The inverse of the matrix* $\boldsymbol{A}$ *is given by* $\boldsymbol{B} = \boldsymbol{A}^{-1}$ *where*

$$\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 6 & 4 \end{pmatrix} \quad \textit{and} \quad \boldsymbol{B} = \begin{pmatrix} 1/4 & 1/8 \\ -3/8 & 1/16 \end{pmatrix}$$

as

$$\boldsymbol{AB} = \begin{pmatrix} 1 & 2 \\ 6 & -4 \end{pmatrix} \begin{pmatrix} 1/4 & 1/8 \\ -3/8 & 1/16 \end{pmatrix} = \boldsymbol{BA} = \begin{pmatrix} 1/4 & 1/8 \\ -3/8 & 1/16 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{I}_2. \qquad \square$$

A Apppendix

Proof of Scalar Tiple Product

To show that

$$\boldsymbol{q} \cdot (\boldsymbol{p} \times \boldsymbol{r}) = -\boldsymbol{p} \cdot (\boldsymbol{q} \times \boldsymbol{r})$$

let $\boldsymbol{s} = \boldsymbol{p} + \boldsymbol{q}$. Now consider

$$\boldsymbol{s} \cdot (\boldsymbol{s} \times \boldsymbol{r}) = \boldsymbol{0}$$

which is zero as the cross product $\boldsymbol{s} \times \boldsymbol{r}$ will be perpendicular to both $\boldsymbol{s}$ and $\boldsymbol{r}$, then the dot product of this vector with $\boldsymbol{s}$ will be zero. Thus,

$$\begin{aligned} \boldsymbol{0} &= (\boldsymbol{p} + \boldsymbol{q}) \cdot ((\boldsymbol{p} + \boldsymbol{q}) \times \boldsymbol{r}) \\ &= (\boldsymbol{p} + \boldsymbol{q}) \cdot (\boldsymbol{p} \times \boldsymbol{r} + \boldsymbol{q} \times \boldsymbol{r}) \\ &= \boldsymbol{p} \cdot (\boldsymbol{p} \times \boldsymbol{r}) + \boldsymbol{p} \cdot (\boldsymbol{q} \times \boldsymbol{r}) + \boldsymbol{q} \cdot (\boldsymbol{p} \times \boldsymbol{r}) + \boldsymbol{q} \cdot (\boldsymbol{q} \times \boldsymbol{r}) \\ &= \boldsymbol{p} \cdot (\boldsymbol{q} \times \boldsymbol{r}) + \boldsymbol{q} \cdot (\boldsymbol{p} \times \boldsymbol{r}) \end{aligned} \tag{26}$$

Hence

$$\boldsymbol{p} \cdot (\boldsymbol{q} \times \boldsymbol{r}) = -\boldsymbol{q} \cdot (\boldsymbol{p} \times \boldsymbol{r})\,. \qquad \blacksquare$$

Proof of Vector Tiple Product

To show that $\boldsymbol{v}$ is in the plane of $\boldsymbol{b}$ and $\boldsymbol{c}$, let the x-axis be parallel to $\boldsymbol{b}$ and $\boldsymbol{c}$ to lie in the xy-plane, then

$$\boldsymbol{a} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}, \quad \boldsymbol{b} = \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{c} = \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix}.$$

Hence $\boldsymbol{b} \times \boldsymbol{c} = (0, 0, b_1 c_2)^T$. Thus

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \begin{pmatrix} a_2 b_1 c_2 \\ -a_1 b_1 c_2 \\ 0 \end{pmatrix}.$$

Now let $d_1 = a_2 b_1 c_2$ and $d_2 = -a_1 b_1 c_2$ then $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{d} = (d_1, d_2, 0)$ which lies in the plane containing $\boldsymbol{b}$ and $\boldsymbol{c}$. To show that $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \lambda \boldsymbol{b} + \mu \boldsymbol{c}$. Multiplying out

$$\begin{aligned} \boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= \begin{pmatrix} a_2 b_1 c_2 - a_1 b_1 c_1 + a_1 b_1 c_1 \\ -a_1 b_1 c_2 \\ 0 \end{pmatrix} \\ &= (a_2 c_2 + a_1 c_1) \begin{pmatrix} b_1 \\ 0 \\ 0 \end{pmatrix} - a_1 b_1 \begin{pmatrix} c_1 \\ c_2 \\ 0 \end{pmatrix} \\ &= (a_2 c_2 + a_1 c_1)\, \boldsymbol{b} - a_1 b_1 \boldsymbol{c} \\ &= (\boldsymbol{a} \cdot \boldsymbol{c})\, \boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\, \boldsymbol{c}. \end{aligned} \qquad \blacksquare$$