

# Integrability, localisation and bifurcation of an elastic conducting rod in a magnetic field

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7<sup>th</sup> Workshop on Dynamical Systems & Ergodic Theory in Northern Germany

June 9<sup>th</sup> 2023

University of Hamburg

Introduction

A rod in a uniform magnetic field

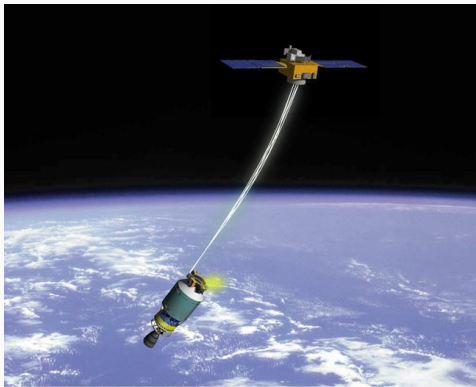
Spatially complex localisation

Hamiltonian-Hopf-Hopf bifurcation

Conclusions

## Introduction

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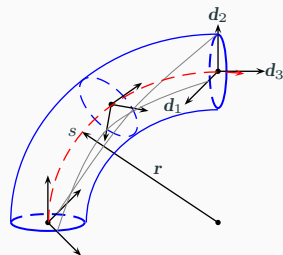


- Classical problem: modern approach
- The reduction estimated at a **billion** dollars over ten years for the international space station alone [1]
- Tethers need to be described as an elastic rod rather wire
- Highly (geometrically) nonlinear

A Cosserat rod is defined by a centreline  $\mathbf{r}$  and directors  $\{\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3\}$  by

$$\mathbf{r}' = \mathbf{v} \quad \text{and} \quad \mathbf{d}'_i = \mathbf{u} \times \mathbf{d}_i.$$

The strains  $\mathbf{u}, \mathbf{v}$  relates the configuration to the balance equations through the hyperelastic constitutive relations. In order to separate geometric from material nonlinearities, linear constitutive relations are chosen throughout

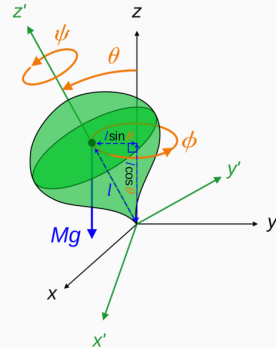


$$u_1 = m_1/B_1, \quad u_2 = m_2/B_2, \quad u_3 = m_3/C \quad \text{and} \quad v_3 = 1 + n_3/K.$$

- Parallels between the **motion** of a heavy spinning top and the **static configurations** of a rod under torque and tension
- Arc-length plays the role of time
- Sleeping top - straight rod
- Precession - helical rod
- Homoclinic - localised modes

but:

- Regions of stability reversed
- Simplification, no shear, no extension, no initial curvature, nonlinear constitutive relations ...



Simplest rod model is the force-free rod. In the spatial frame

$$\mathbf{m}' = \mathbf{0}.$$

In the director frame

$$\mathbf{m}' = \mathcal{J}(\mathbf{m}) \nabla \mathcal{H}(\mathbf{m}) = \mathbf{m} \times \mathbf{u}$$

structure matrix and Hamiltonian given by

$$\mathcal{J} = -\mathcal{J}^T = \hat{\mathbf{m}} = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{H} = \frac{1}{2} \mathbf{m} \cdot \mathbf{u}(\mathbf{m}).$$

One-dimensional null-space spanned by the gradient of the Casimir

$$\nabla C_1 = \frac{1}{2} (m_1, m_2, m_3)^T \Rightarrow C_1 = \mathbf{m} \cdot \mathbf{m}.$$

The system is globally **superintegrable** [2].

The non-canonical system is in fact a Lie-Poisson system

$$\{f, g\}_\mu = \left\langle \mu, \left[ \frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle$$

where  $\mu \in \mathfrak{g}^*$  and  $f, g: \mathfrak{g}^* \mapsto \mathbb{R}$ , with an associated inner product  $\langle \cdot, \cdot \rangle$  and Lie bracket on the Lie algebra  $[\cdot, \cdot]$ .

The inner product associates elements of the Lie algebra with its dual and the Lie bracket acts on the function derivatives of  $f$  and  $g$  with respect to the field variable, that is  $\delta f / \delta \mu: \mathfrak{g}^* \mapsto \mathfrak{g}$ .



The Poisson bracket is a Lie-Poisson bracket

$$\{f, g\}_{(m)} = - \underbrace{m \cdot (\nabla_m f \times \nabla_m g)}_{\text{twist}}.$$

where the Lie bracket is given by the direct sum of elements

$$[\xi, \eta] = \xi \times \eta$$

and the inner product is the dot product.

Corresponds to the Euler top.

For a rod under end tension and moment in the spatial frame

$$\mathbf{m}' = \mathbf{n} \times \mathbf{r}' \quad \text{and} \quad \mathbf{n}' = \mathbf{0}.$$

In the body frame

$$\begin{pmatrix} \mathbf{m} \\ \mathbf{n} \end{pmatrix}' = \begin{pmatrix} \hat{m} & \hat{n} \\ \hat{n} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}, \quad \mathcal{H} = \frac{1}{2} \mathbf{u} \cdot \mathbf{m} + \frac{1}{2} (\mathbf{v} - \mathbf{d}_3) \cdot \mathbf{n} + \mathbf{n} \cdot \mathbf{d}_3$$

where  $\mathbf{d}_3 = (0, 0, 1)$ . System has two Casimirs

$$C_1 = \mathbf{n} \cdot \mathbf{n} \quad \text{and} \quad C_2 = \mathbf{n} \cdot \mathbf{m}.$$

If the principal bending stiffnesses are equal, i.e. isotropic, is **integrable**.

$$I_1 = \mathbf{m} \cdot \mathbf{d}_3.$$

Other integrable cases: Kovalevskaya and Chaplygin-Goryachev.

The force is fixed in the spatial frame, but rotates in the body frame. The dynamics are generated by a Poisson bracket with a semi-direct product extension

$$\{f, g\}_{(m,n)} = -m \cdot (\nabla_m f \times \nabla_m g) - \underbrace{n \cdot (\nabla_m f \times \nabla_n g + \nabla_n f \times \nabla_m g)}_{\text{force}}.$$

which is a Lie-Poisson bracket for

$$[(\xi, u), (\eta, v)] = (\xi \times \eta, \xi \times v - \eta \times u).$$

Corresponds to a heavy top

Superintegrable solutions are helices with additional integral  $m \cdot m$ .

## Integrability

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For a conducting rod in a uniform magnetic field, in the spatial frame ( $\mathbf{B} = B\mathbf{e}_3$ )

$$\mathbf{m}' = \mathbf{n} \times \mathbf{r}', \quad \mathbf{n}' = \underbrace{\lambda \mathbf{e}_3 \times \mathbf{r}'}_{\text{Lorentz force, } \lambda = IB} \quad \text{and} \quad \mathbf{e}_3' = \mathbf{0}.$$

In the director frame - noncanonical Hamiltonian system

$$\begin{pmatrix} \mathbf{m} \\ \mathbf{n} \\ \mathbf{e}_3 \end{pmatrix}' = \begin{pmatrix} \hat{m} & \hat{n} & \hat{e}_3 \\ \hat{n} & \lambda \hat{e}_3 & 0 \\ \hat{e}_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \\ 0 \end{pmatrix}, \quad \mathcal{H} = \frac{1}{2} \mathbf{u} \cdot \mathbf{m} + \frac{1}{2} (\mathbf{v} - \mathbf{d}_3) \cdot \mathbf{n} + \mathbf{d}_3 \cdot \mathbf{n}.$$

The system has a Poisson bracket:

$$\begin{aligned} \{f, g\}_{(m,n,e_3)} = & -m \cdot (\nabla_m f \times \nabla_m g) - n \cdot (\nabla_m f \times \nabla_n g + \nabla_n f \times \nabla_m g) \\ & - \underbrace{e_3 \cdot (\nabla_m f \times \nabla_{e_3} g + \nabla_{e_3} f \times \nabla_m g)}_{\text{evolution of field}} \\ & - \underbrace{\lambda e_3 \cdot (\nabla_n f \times \nabla_n g)}_{\text{effect of field}} \end{aligned}$$

extended by **semi-direct extension**, describing the evolution of the magnetic field in the body, and a cocycle, describing the Lorentz force, called a **Liebnitz extension** [3]. Lie bracket is

$$[(\xi, u, w), (\eta, v, x)] = (\xi \times \eta, \xi \times v - \eta \times u, \xi \times x - \eta \times w - \lambda u \times v).$$

A nine-dimensional system with three Casimirs

$$C_1 = e_3 \cdot e_3, \quad C_2 = e_3 \cdot n \quad \text{and} \quad C_3 = n \cdot n + 2\lambda m \cdot e_3,$$

two additional first integrals when isotropic and inextensible

$$I_1 = m \cdot d_3 \quad \text{and} \quad I_2 = n \cdot m + \lambda B_1 d_3 \cdot e_3$$

and one Hamiltonian

$$\mathcal{H} = \frac{1}{2} u \cdot m + d_3 \cdot n.$$

Thus system is completely integrable

Conserved quantities have no immediate physical interpretation.

The family of equations can be generated by a **Lax pair** [4] with a spectral parameter  $\alpha$

$$\frac{d}{ds}\Gamma(\alpha) = \left[ \Gamma(\alpha), \hat{d}_3\alpha + \hat{u} \right],$$

where

$$\Gamma(\alpha) = K\hat{d}_3\alpha + \Gamma_0 + \Gamma_1\alpha^{-1} + \dots + \Gamma_n\alpha^{-n} \in \mathfrak{so}(3), \quad n \in \mathbb{N},$$

with

$$\Gamma_0 = \hat{m}, \quad \Gamma_1 = \hat{n} \quad \text{and} \quad \Gamma_2 = \lambda\hat{e}_3 \dots$$

and conserved quantities

$$I_i = -\frac{1}{4} \text{residue}_{\mu=0} \left( \alpha^{i-1} \text{trace} \left[ \Gamma(\alpha)^2 \right] \right) \quad \text{for } i = -1, 0, 1, \dots, n-1,$$
$$C_i = -\frac{1}{4} \text{residue}_{\mu=0} \left( \alpha^{i-1} \text{trace} \left[ \Gamma(\alpha)^2 \right] \right) \quad \text{for } i = n, n+1, n+2, \dots, 2n.$$



Thus the governing equations for elastic conducting rod in a **non-uniform** “twisted” hyper-magnetic field are integrable.

$$\mathbf{m}' + \mathbf{r}' \times \mathbf{n} = \mathbf{0}, \quad \mathbf{n}' + \mathbf{r}' \times \mathbf{B} = \mathbf{0}, \quad \mathbf{B}' + \mathbf{r}' \times \mathbf{D} = \mathbf{0} \quad \text{and} \quad \mathbf{D}' = \mathbf{0}.$$

Gives  $B_x = y, B_y = -x, B_z = 0$ , where  $(x, y, z)$  and  $(B_x, B_y, B_z)$  are components of  $\mathbf{r}$  and  $\mathbf{B}$  relative to the spatial frame  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , and  $\mathbf{e}_3$  is chosen to be in the direction of  $\mathbf{D}$ . Thus the system describes a rod in a linearly-varying magnetic field generated by a uniform ‘hypermagnetic’ field  $\mathbf{D}$ . Hamiltonian is as before, structure matrix is

$$\mathcal{J} = \begin{pmatrix} \hat{m} & \hat{n} & \hat{B} & \hat{D} \\ \hat{n} & \hat{B} & \hat{D} & 0 \\ \hat{B} & \hat{D} & 0 & 0 \\ \hat{D} & 0 & 0 & 0 \end{pmatrix}.$$

## Spatial Chaos

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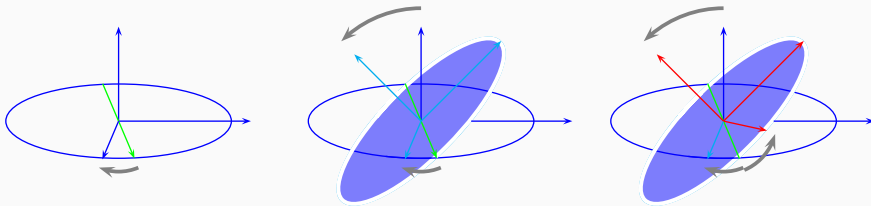


Figure 1: The Euler angles relating two coordinate frames.

Euler angles reduce nine-dimensional non-canonical system to six-dimensional canonical system

$$\mathcal{H}(\theta, p_\theta, \psi, p_\psi) = \frac{p_\theta^2}{2B_1} + \frac{1}{2B_1} \left( \frac{p_\psi - p_\phi \cos \theta}{\sin \theta} \right)^2 + \frac{p_\phi^2}{2C} + C_2 \cos \theta + \sin \psi \sin \theta \sqrt{C_3 - C_2^2 - 2\lambda p_\psi}$$

with two integrals

$$I_1 = p_\phi, \quad I_2 = B_1 \lambda \cos \theta + C_2 p_\psi - \sqrt{C_3 - C_2^2 - 2\lambda p_\psi} \left( p_\theta \sin \psi - \cos \psi \left( \frac{p_\phi - p_\psi \cos \theta}{\sin \theta} \right) \right).$$

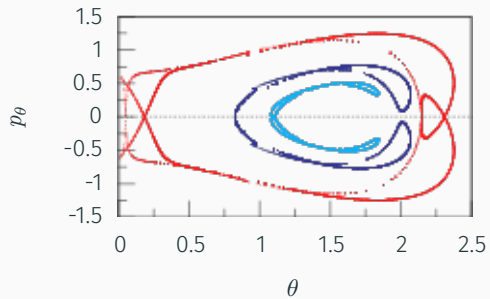
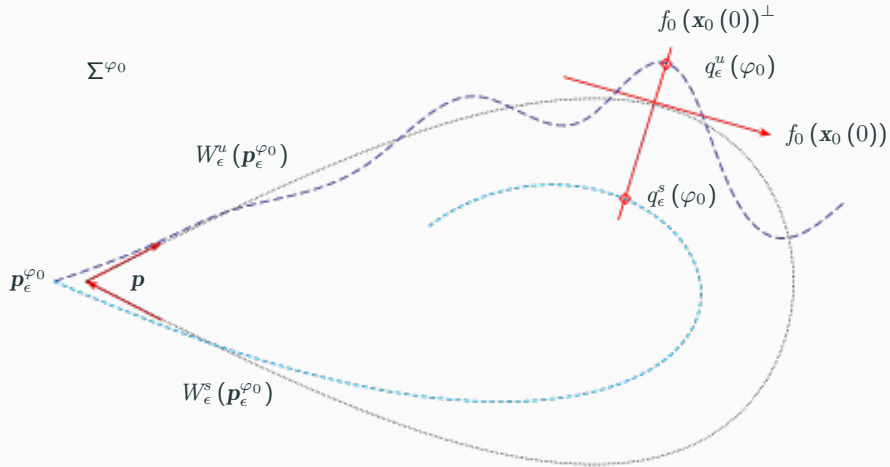


Figure 2: Phase plane at various energy levels at section  $\cos \psi = 0.9$ .

An approximation to splitting of stable/unstable manifolds



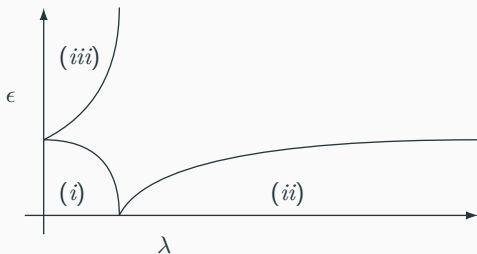
On the scaling

$$2C_1 - C_2^2 = a\delta^2, \quad \lambda = b\delta^2 \quad \text{and} \quad C_2/K = c\delta$$

reduced extensible Hamiltonian takes the form

$$\begin{aligned} \mathcal{H}_\delta(\theta, p_\theta, \psi, p_\psi) = & \mathcal{H}_0(\theta, p_\theta) + \delta \left( \mathcal{H}_1^\lambda(\theta, \psi, p_\psi) + \mathcal{H}_1^\epsilon(\theta) \right) \\ & + \delta^2 \mathcal{H}_2^{\epsilon\lambda}(\theta, \psi, p_\psi) + \delta^3 \mathcal{H}_3^{\epsilon\lambda^2}(\theta, \psi, p_\psi) + \mathcal{O}(\delta^4). \end{aligned}$$

Mel'nikov's method can be applied in three possible cases:



- Perturb Lagrange rod,
- Perturb extensible rod,
- Perturb magnetic rod.

If extensible detailed Mel'nikov analysis shows loss of integrability.

$$\begin{aligned}
 \mathcal{M}_h^{(1)}(\psi_0) &= \int_{-\infty}^{+\infty} f_0 \wedge f_1 d\psi = \int_{-\infty}^{+\infty} \left\{ \mathcal{H}_0 + \mathcal{H}_1^\epsilon, \mathcal{H}_1^\lambda \right\}_{(\theta, p_\theta)} ds \\
 &= 2\sqrt{a - 2bp_\psi} \cos \psi_0 \int_0^{+\infty} p_\theta \cos \bar{\psi} \sin \theta \left( 1 + \frac{C_2}{K} \cos \theta \right) ds \\
 &\quad - 2\sqrt{a - 2bp_\psi} \sin \psi_0 \int_0^{+\infty} p_\theta \sin \bar{\psi} (1 + \cos \theta) \left( \cos \theta + \frac{C_2}{K} \cos 2\theta \right) ds.
 \end{aligned}$$

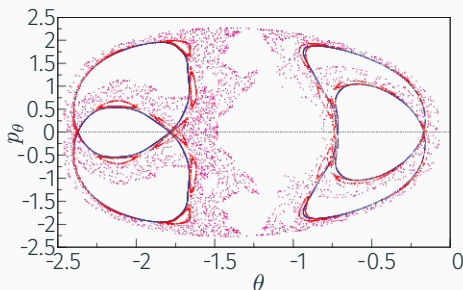


Figure 3: Poincaré sections

For case (ii)

$$\mathcal{M}_h^{(1)}(\psi_0) = \int_{-\infty}^{+\infty} \left\{ \mathcal{H}_0, \mathcal{H}_1^\epsilon + \mathcal{H}_1^\lambda \right\}_{(\theta, p_\theta)} ds$$

but both perturbations are integrable

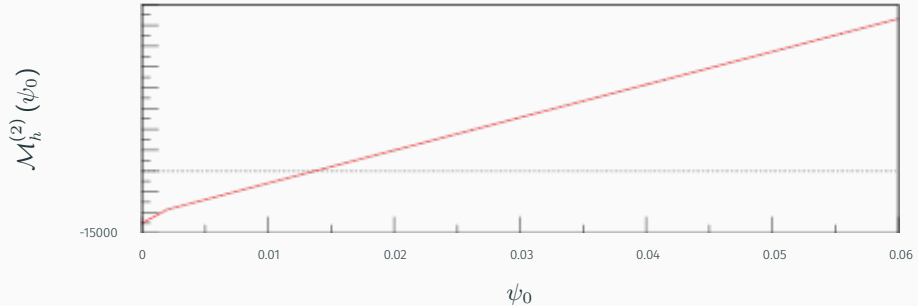
$$\int_{-\infty}^{+\infty} \left\{ \mathcal{H}_0, \mathcal{H}_1^\epsilon \right\}_{(\theta, p_\theta)} ds = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \left\{ \mathcal{H}_0, \mathcal{H}_1^\lambda \right\}_{(\theta, p_\theta)} ds = 0.$$

Thus **second order** analysis needs to be performed [5].

$$\begin{aligned} \mathcal{M}_h^{(2)}(\psi_0) = & \frac{1}{2} \int_{\psi_0}^{+\infty} f_0 \wedge D^2 f_0(\mathbf{x}_1^s)^2 d\psi + \frac{1}{2} \int_{-\infty}^{\psi_0} f_0 \wedge D^2 f_0(\mathbf{x}_1^u)^2 d\psi \\ & + \frac{1}{2} \int_{\psi_0}^{+\infty} f_0 \wedge Df_1 \mathbf{x}_1^s d\psi + \frac{1}{2} \int_{-\infty}^{\psi_0} f_0 \wedge Df_1 \mathbf{x}_1^u d\psi \\ & + \int_{-\infty}^{+\infty} f_0(\mathbf{x}_0) \wedge f_2(\mathbf{x}_0, \psi) d\psi. \end{aligned}$$



- Homoclinic solutions found numerically with angle  $\psi$  as 'time' and action  $\mathcal{I}$  as Hamiltonian.
- Robust numerical method using bisection method to find first order approximation to flow,  $\mathbf{x}_1$  computed subject to being **bounded** and **transverse** to flow forwards and backwards in time.



**Figure 4:** Second order Mel'nikov integral showing existence of simple zero

## Bifurcation

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Ordinarily trivial configuration, a straight twisted rod, is about a fixed point, now it is a **periodic** solution  $\gamma(s)$  with period  $\tau$ .

$$\gamma(s) = (0, 0, 0, 0, 0, 0, 0, 0, \sin(s(1 + \nu)/2), \cos(s(1 + \nu)/2)).$$

Under the  $\tau$ -mapping  $\gamma(\tau) = \mathbf{p}$  the periodic orbit is the fixed point

$$\mathbf{p} = \gamma(\tau) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

- Computation and continuation of localised solutions needs modification.
- Floquet theory required, monodromy matrix computed.
- No closed form analytical expressions for buckling values.

The system is **reversible**. Thus two involutions exist

$$\begin{aligned} R_1 : (m_1, m_2, m_3, n_1, n_2, n_3, e_{31}, e_{32}, e_{33}) &\mapsto \\ &(-m_1, m_2, m_3, -n_1, n_2, n_3, e_{31}, -e_{32}, e_{33}) \quad \text{and} \\ R_2 : (m_1, m_2, m_3, n_1, n_2, n_3, e_{31}, e_{32}, e_{33}) &\mapsto \\ &(m_1, -m_2, m_3, n_1, -n_2, n_3, e_{31}, -e_{32}, e_{33}) \quad \text{as } s \mapsto -s. \end{aligned}$$

$R_i$ -Reversible configurations pass through a **symmetric section**  $\mathcal{S}_i$

$$\mathcal{S}_i = \text{Fix}(R_i) \quad \text{for } \gamma(n\tau) = \mathbf{p} \quad \text{where with } \mathbf{p} \in \mathcal{S}_i.$$

Symmetric section is three-dimensional

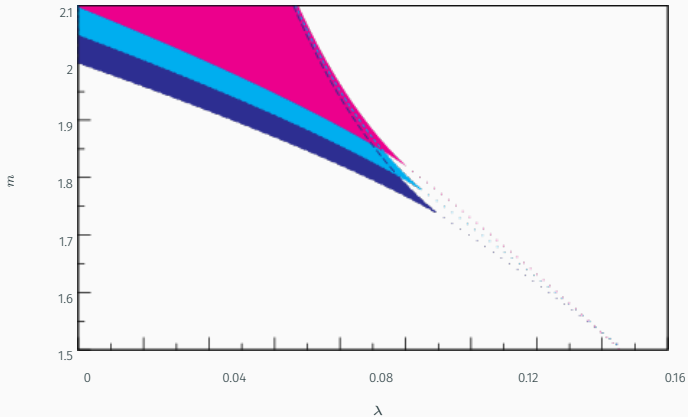
$$\mathcal{S}_1 = \left\{ \mathbf{x} \in \mathbb{R}^9 : m_1(1) = n_1(1) = e_{31}(1) = 0 \right\}.$$

- In order to avoid the polar singularity inherent in the Euler angles when  $\theta = 0$  and reduce the dimension of the full system Euler parameters are used to convert quantities from the spatial frame into the director frame.

- The four Euler parameters  $q = (q_1, q_2, q_3, q_4)$  are subject to

$$1 = q_1^2 + q_2^2 + q_3^2 + q_4^2.$$

- The ten-dimensional system  $\mathbf{x} = (m, n, q)$  has three Casimirs and a constraint.

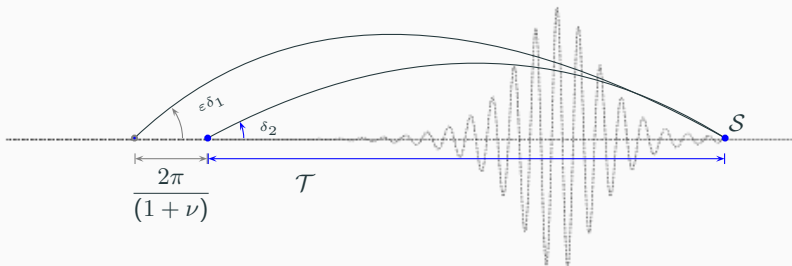


**Figure 5:** Spectrum of the monodromy matrix when  $\nu = 1/3$  with  $\epsilon = 0$  (blue),  $\epsilon = 0.05$  (cyan) and  $\epsilon = 0.1$  (magenta). The coloured regions correspond elliptic periodic orbits, the dashed lines are codimension-one curves at which the multipliers are stationary and reverse direction.

- Shooting from fixed point of a map  $\mathbf{p}$  over half range into 3D symmetric section  $\mathcal{S}$  of a reversibility.
- Three parameter shooting with  $(\delta_1, \delta_2, \mathcal{T})$

$$\mathbf{x}(0) = \mathbf{p} + \varepsilon \delta_1 (\mathbf{v}_1 \sin \delta_2 + \mathbf{v}_2 \cos \delta_2)$$

where  $\mathbf{v}_1$  and  $\mathbf{v}_2$  are vectors in the two-dimensional unstable linear subspace of the fixed point  $\mathbf{p}$ .



- Homoclinic solutions are **codimension-zero**.

Localised solutions are found for isolated branches of the shooting parameters.

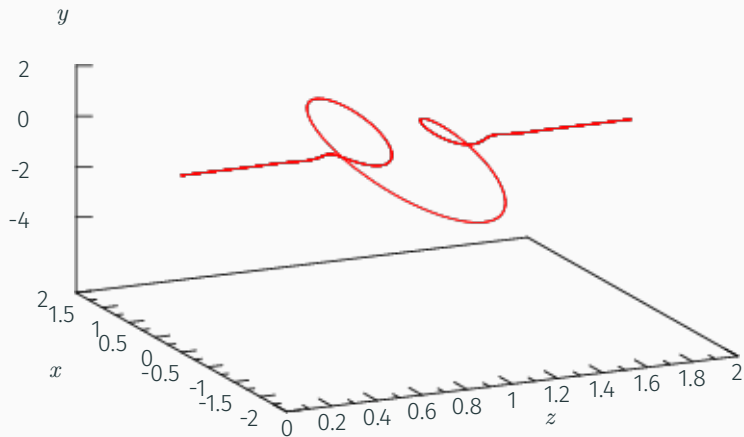


Figure 6: A localised solution



When extensible or anisotropic there are an infinite number of multi-modal solutions comprised of “primary” uni-modal solutions.

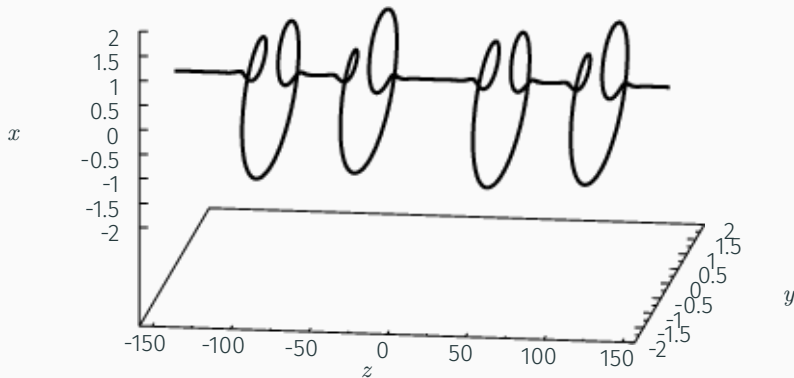


Figure 7: Localised multi-pulse solution

With Euler parameters 10D monodromy matrix decouples:

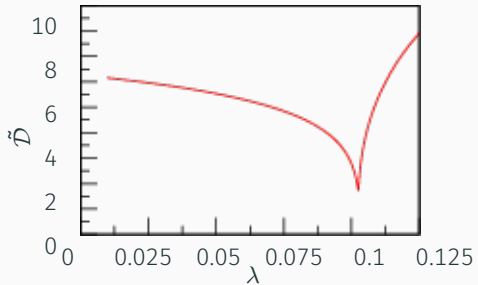
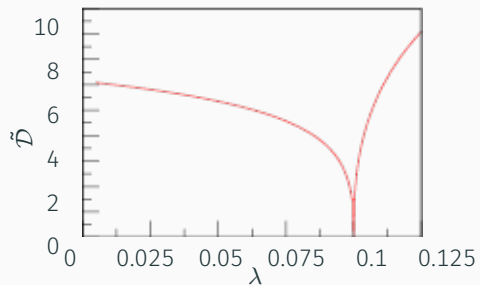
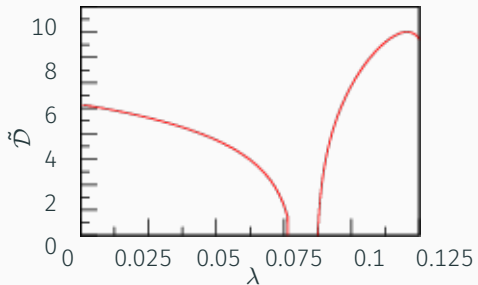
- 4D trivial matrix containing Casimirs and constraint,
- 6D nontrivial matrix contains the 'dynamics'.

Projection boundary conditions using `auto97` exploiting **exponential trichotomies** onto two-dimensional stable and centre manifolds of fixed point of periodic orbit [6]

$$(L_{c,s} - \mathbf{p}) \mathbf{x} = \mathbf{0} \quad \text{where} \quad L_{c,s} \in \mathbb{R}^{6 \times 2}$$

along with conditions on three-dimensional symmetric section.

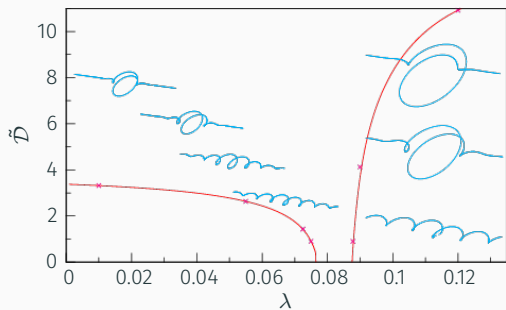
- **Ill-posed**: an extra boundary condition  $\Rightarrow$  truncation length  $\mathcal{T}$  freed.



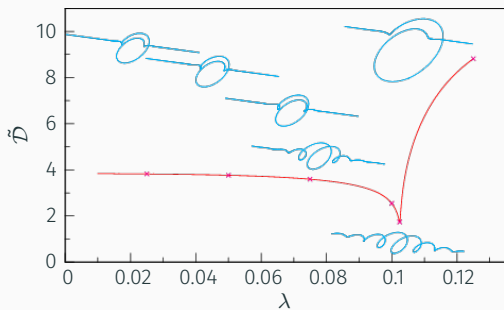
Hamiltonian-Hopf bifurcations at

- two  $\lambda_{\pm}$  when  $m > m_c$ ,
- one  $\lambda_c$  when  $m = m_c$ ,
- zero when  $m < m_c$ ,

critical values.



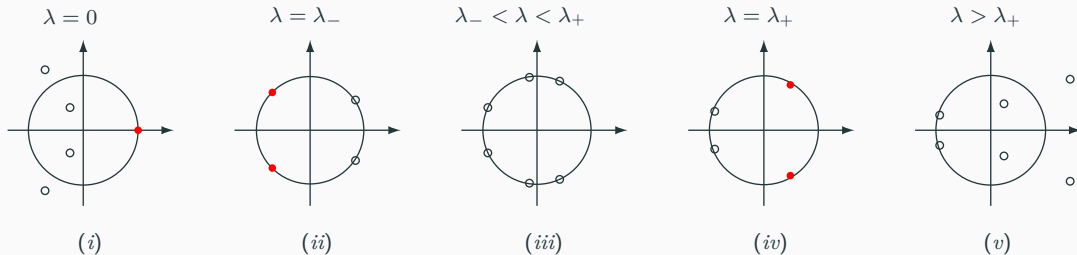
(a)  $m = 1.9$



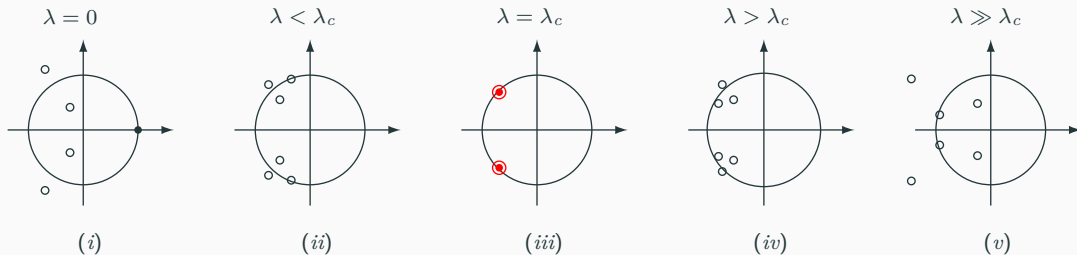
(b)  $m = 1.81$

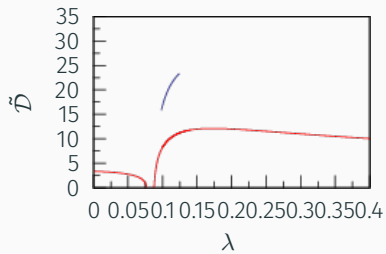
- Hamiltonian-Hopf bifurcation about a periodic solution from right  $\lambda = \lambda_-$  and left  $\lambda = \lambda_+$ . Localised solutions bifurcate into straight twisted rods.
- Localisation-delocalisation-localisation can occur near codimension-two point.

Behaviour of Floquet multipliers for  $m > m_c$

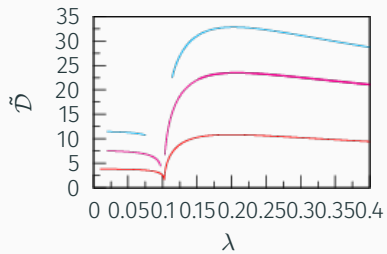


and  $m = m_c$ .

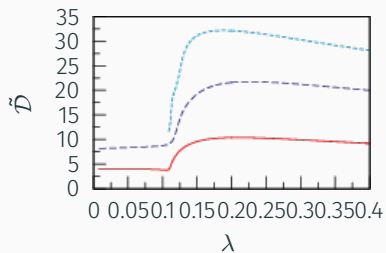




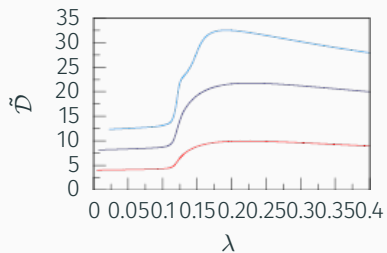
(a)  $m = 1.9$



(b)  $m = 1.81$



(c)  $m = 1.775$



(d)  $m = 1.3985$

**Figure 9:** Bifurcation diagram for single and multimodal localised configurations

## Summary

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- System is **Hamiltonian**, **Lie-Poisson** and **completely integrable**
- Physical realisation of abstract system [3]
- System is one member in a family of integrable systems expressed by a **Lax pair** [4]
- Hidden conditions on integrability: extensibility, shearability ...



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- If extensible non-integrable and has spatially chaotic solutions
  
- Three parameter shooting, exponential trichotomies
- **Hamiltonian-Hopf-Hopf** bifurcation
- Buckling due to increasing and decreasing external force
- **Sequential** merging of limit points for multimodal solutions

- Complete reduction to single degree of freedom system [7] may enable
  - Expression of system as an equivalent oscillator
  - Fixed point solutions: existence of superintegrable solutions
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- Normal form for Hamiltonian-Hopf-Hopf bifurcation in the twistless case

Thankyou for your attention.

Any questions?



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github.com/djps/extensibility



djps.github.io

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