The Buckling of Magneto-Strictive Cosserat Rods

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Summary. Magneto-striction is a property of ferromagnetic materials which causes them to change their shape or dimensions during the process of magnetization. A conducting ferromagnetic rod in a magnetic field will experience a Lorentz body force and change size, coupling the effective material properties to the loading conditions. Homoclinic solutions, relating to localised post-buckled configurations, and the post-buckling curves are computed, illustrating the influence of magnetostriction the buckling behaviour.

Introduction

The buckling of an elastic conducting body in a magnetic field is a classical problem, but recently has received attention as a model for electrodynamic space tethers, i.e., long conducting cables that exploit the Earth's magnetic field to generate thrust and drag forces for manoeuvring while de-orbiting. In this brief paper the effects of magnetostriction are investigated. Magnetostriction was first documented by Joule in 1842 [1] and today magnetostrictive materials play an important in ultrasonic transducers, actuators and sensors [2]. The changes in volume associated with magnetostriction may be a small fraction of the overall volume, but the consequences will be significant through the loss of integrability and the consequent appearance of multi-modal localised configurations [3].

Magneto-Striction Cosserat Rods

The geometrically exact formulation of an isotropic conducting elastic rod under end force and moment and Lorentz body force leads to the static force and moment balance equations coupled to the direction of the magnetic field in the body frame. The system is Hamiltonian, and in the absence of either the Lorentz force or extensibility, is completely integrable.

$$
\begin{aligned}
x_1' &= (1+\nu)\,x_2x_6 - x_3x_5 + 2\lambda\,(1+\gamma x_3)\,(x_7x_{10} + x_8x_9)\,, & x_6' &= 0,\\
x_2' &= x_3x_4 - (1+\nu)\,x_1x_6 - 2\lambda\,(1+\gamma x_3)\,(x_7x_9 - x_8x_{10})\,, & x_7' &= (x_4x_{10} - x_5x_9 + (1+\nu)\,x_6x_8)/2,\\
x_3' &= x_1x_5 - x_2x_4, & x_8' &= (x_4x_9 + x_5x_{10} - (1+\nu)\,x_6x_7)/2,\\
x_4' &= \nu x_5x_6 + x_2\,(1+\gamma x_3)/m^2, & x_9' &= (-x_4x_8 + x_5x_7 + (1+\nu)\,x_6x_{10})/2\\
x_5' &= -\nu x_4x_6 - x_1\,(1+\gamma x_3)/m^2, & x_{10}' &= (-x_4x_7 - x_5x_8 - (1+\nu)\,x_6x_9)/2.
\end{aligned}
$$

For a complete derivation, and discussion of the symmetries of the system, see [3]. The system has a periodic solution

$$
\boldsymbol{p}(t) = (0,0,1,0,0,1,0,0,\cos(t\,(1+\nu)/2)\,, -\sin(t\,(1+\nu)/2))\,,
$$

corresponding to a straight but loaded rod. There are two non-dimensional loading parameters; λ the magnetic body force and m the end force, and two nondimensional material parameters; γ the extensibility and ν the torsional stiffness. As a illustrative model of an magnetostrictive material, the dependence of the extensibility on the magnetic field is given by

$$
\gamma(\lambda) = \gamma_0 + \alpha\left(\tan^{-1}(\beta\lambda)\right)^2 \quad \text{with} \quad \alpha = \frac{\pi^2}{4}(\gamma_\infty - \gamma_0) \quad \text{so that} \quad \frac{\mathrm{d}\gamma}{\mathrm{d}\lambda} = \frac{2\alpha\beta\tan^{-1}(\beta\lambda)}{1+\beta^2\lambda^2}
$$

where γ_0 is the extensibility in the absence of the magnetic field, γ_∞ the saturation extensibility, and β is related to the derivative of extensibility. It is assumed that the material properties change instantaneously as the magnetic field strength is increased. An example of the magneto-strictive realtionship is displayed in Fig 1.

Linear Analysis

Numerically computed non-trivial Floquet multiplier configurations of the periodic solution in the (λ, m) parameter plane are shown in Fig. 2. Within the shaded region which terminates in a cusp, all multipliers lie on the unit circle.Outside

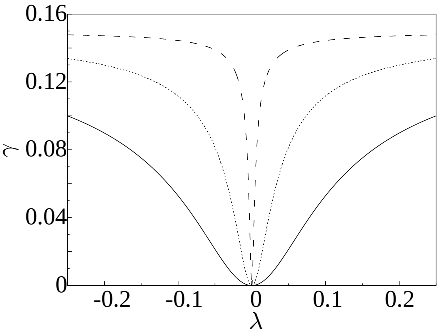


Figure 1: Extensibility against magnetic field strength where $\gamma_0 = 0$ and $\alpha = 3/(5\pi^2)$ with $\beta = 13.6525\tan(\pi/\sqrt{6})$ (solid), $\beta = 20\tan(\pi/\sqrt{6})$ (dot) and $\beta = 100\tan(\pi/\sqrt{6})$ (dash).

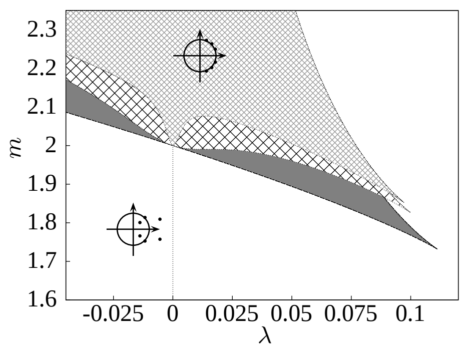


Figure 2: Spectrum of Floquet multipliers when $\nu = 1/3$ for the degrees of extensibility given in Fig. 1.

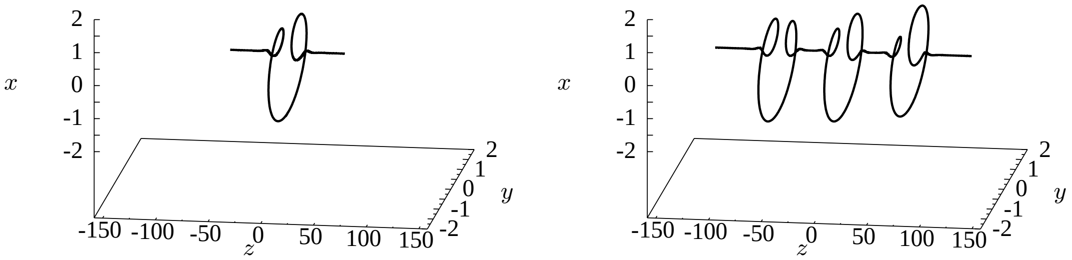


Figure 3: Single and multi-modal solutions for $\lambda = 0.05$, $m = 1.9$, $\nu = 1/3$ and $\gamma(0.05) = 1/10$.

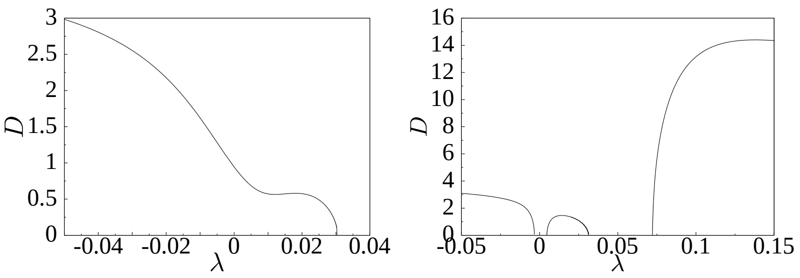
this region there are a pair of multipliers inside, on and outside the unit circle. The boundaries of this region corresponds to Hamiltonian-Hopf bifurcations in which two pairs of multipliers collide on the unit circle. At the cusp, where the boundaries merge, the Hamiltonian-Hopf bifurcations merge in a Hamiltonian-Hopf-Hopf bifurcation. The location of the cusp is primarily determined by saturation extensibility. For all rods with $\gamma_0 = 0$, when $\lambda = 0$ the rods buckle at the classical Timoshenko value $m = 2$. However, for rods with large α and β, i.e. those which quickly reach a saturation value which is significantly different from the intrinic extensibility, the Timoshenko load may be a smooth local minima and there will be an additional pair of buckling values.

Computation & Continuation of Post-Buckled Configurations

The post-buckled solutions were computed using a shooting method exploiting the reversibilities of the system. From the linear analysis, a good initial guess was computed, placing the initial condition on the two-dimensional linearised flow of the unstable manifold near the saddle point. A variational problem was then solved for three shooting parameters δ, θ and $\mathcal{T}$, so that the solutions would intersect a three-dimensional symmetric section over the half range $\mathcal{T}$

$$\boldsymbol{x}(0) = \boldsymbol{p}(0) + \varepsilon\delta\left(\boldsymbol{v}_1 \sin\theta + \boldsymbol{v}_2 \cos\theta\right) \quad \text{such that} \quad x_1(\mathcal{T}) = 0, \quad x_4(\mathcal{T}) = 0 \quad \text{and} \quad x_7(\mathcal{T}) = 0.$$

The configurations were continued using the symmetric section to satisfy the right hand conditions and projection boundary conditions to satisfy the left hand conditions. The solutions were projected onto the linearised centre and unstable eigenspaces. The eigenspaces share a common direction, thus the system was over-determined. To resolve this the truncation length $\mathcal{T}$ was allowed to vary with the principle bifurcation parameter. The end shortening, D, which measures the change in computed length of the buckled rod from that of a straight rod was computed in order to qualitatively investigate the behaviour of the system.



(a) $\beta = 13.6525 \tan \frac{\pi}{\sqrt{6}}$ (b) $\beta = 100 \tan \frac{\pi}{\sqrt{6}}$

Figure 4: Load-deflection curves for a primary orbit when $m = 2.05$ with $\alpha = 3/(5\pi^2)$.

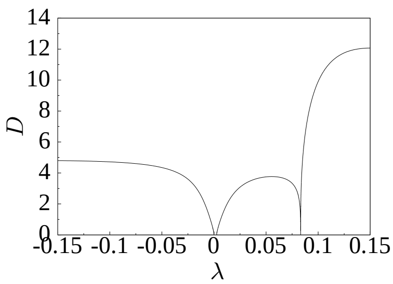


Figure 5: Two codimension-two bifurcations occuring when $m = 2$, $\alpha = 0.17064$ and $\beta = 337.2634$.

Discussion

The post-buckling behaviour is determined by the change in material properties. As illustrated in Figs. 1 and 2, there are essentailly two situations: when the change in material properties is weak or strong. In the case of weak change the post-buckling behaviour is essentially determined by the cusp. For strongly magnetostrictive rods the post-buckling behaviour is influenced by both the behaviour about the Timoshenko load, determined by γ_0 and the cusp, determined by γ_∞, as illustrated by Fig 4(b). In the transition from weakly to strongly magnetostriction there are regimes in which an increase in the body force, which typically lowers the buckling threshold, is matched by an increase in the extensibility, which increases the buckling threshold. Fig 4(a) illustrates this phenomena: the load-deflection characteristics are approximately stationary for a small range of λ. As illustrated in Fig. by determining the value at which the Timoshenko point is a codimension-two point, the location of the cusp can then be varied so that both codimension-two points exist. Note that the load-deflection curve is significantly steeper about the cusp than the Timoshenko point.

[1] Joule J. P (1847), On the effects of magnetism upon the dimensions of iron and steel bars, *London, Edinburgh Dublin Phil. Mag. J. Sci* **33**:76-87.

[2] Dobbs E. R. (1973), Electromagnetic Generation of Ultrasonic Waves, *Physical Acoustics; Principles and Methods* Academic Press, NY

[3] Sinden D. and van der Heijden G. H. M. (2009), Spatial chaos of an extensible elastic rod in a uniform magnetic field, *J. Phys. A: Math. Theor.* **42**:375207.