Localisation of a twisted conducting rod in a uniform magnetic field: the Hamiltonian-Hopf-Hopf bifurcation

G.H.M. van der Heijden, D. Sinden
Centre for Nonlinear Dynamics, Department of Civil, Environmental & Geomatic Engineering, University College London, Gower Street, London WC1E 6BT, United Kingdom

Summary. We study localised (multi-pulse homoclinic) post-buckling solutions for an extensible conducting rod under end loads and placed in a uniform magnetic field. The homoclinic bifurcation behaviour is found to be organised by a codimension-two Hamiltonian-Hopf-Hopf bifurcation. We predict new stability results for twisted magnetic rods which are relevant for electrodynamic space tethers and potentially for conducting nanowires in future electromechanical devices.

Introduction

The buckling of an elastic conducting wire in a magnetic field is a classical problem in magnetoelasticity. Wolfe [1] gave the first rigorous bifurcation analysis in the case of a uniform magnetic field directed parallel to the undeformed wire (modelled as a rod). He considered bifurcation from a straight and untwisted rod, which is described by a fixed point of the equilibrium equations. He did not consider post-buckling behaviour. Here, by allowing for end loads, we consider buckling from a straight but twisted state. This state is described by a periodic solution of the equations and therefore complicates the analysis.

A Hamiltonian formulation of the magnetic rod problem was given in [2] where it was shown that for a transversely isotropic inextensible and unshearable rod the equilibrium equations are completely integrable, while in a follow-up paper [3] it was shown that if the inextensibility/unshearability assumption is dropped the equations become nonintegrable and give rise to complicated (chaotic) dynamics, including multi-pulse homoclinic orbits. Here we investigate the existence, multiplicity and bifurcation of these homoclinic orbits. The existence of conserved quantities makes the trivial periodic solution non-hyperbolic requiring previous numerical (shooting) techniques to be adapted.

The study of a conducting rod in a magnetic field is of interest for stability problems in electrodynamic space tethers, i.e., conducting cables that exploit the earth's magnetic field to generate thrust and drag (Lorentz) forces for manoeuvring. If long tethers, tensioned by the earth's gravity gradient, connect large end masses then any twist as a result of relative motion of these masses in combination with the magnetic force may induce buckling.

Equilibrium equations

We use the geometrically-exact Cosserat theory in which a rod is characterised by a space curve $\boldsymbol{r}(s)$, describing the centreline of the rod, and an attached right-handed orthonormal triad of directors $\{\boldsymbol{d}_1(s), \boldsymbol{d}_2(s), \boldsymbol{d}_3(s)\}$ describing the varying orientation of the cross-section [2]. Here s is an arbitrary parameter.

The equilibrium equations for the internal force $\boldsymbol{n}$ and moment $\boldsymbol{m}$ (assumed averaged over the cross-section of the rod) are (using a prime to denote differentiation with respect to s)

$$\boldsymbol{n}' + \boldsymbol{f} = \boldsymbol{0}, \qquad \boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} = \boldsymbol{0}. \tag{1}$$

The external distributed load $\boldsymbol{f}$ is here given by the Lorentz force induced by a uniform magnetic field $\bar{\boldsymbol{B}} = \bar{B}\boldsymbol{e}_3$ if the rod carries an electric current $\boldsymbol{I} = I\boldsymbol{r}'$, i.e., $\boldsymbol{f} = \boldsymbol{I} \times \bar{\boldsymbol{B}} = I\bar{B}\,\boldsymbol{r}' \times \boldsymbol{e}_3$.

We parametrise rotations of the director frame $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ relative to the fixed frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ by means of so-called Euler parameters, i.e., by a quadruple of real numbers $\boldsymbol{q} = (q_1, q_2, q_3, q_4)$ subject to the norm condition $\boldsymbol{q} \cdot \boldsymbol{q} = 1$. For these parameters we have the following equation in terms of the bending and twisting strains $u_i = \frac{1}{2}\varepsilon_{ijk}\boldsymbol{d}_j' \cdot \boldsymbol{d}_k$:

$$\boldsymbol{q}' = \frac{1}{2}A^T \mathsf{u}, \quad \text{where} \quad \mathsf{u} = (u_1, u_2, u_3)^T \quad \text{and} \quad A = \begin{pmatrix} q_4 & q_3 & -q_2 & -q_1 \\ -q_3 & q_4 & q_1 & -q_2 \\ q_2 & -q_1 & q_4 & -q_3 \end{pmatrix}. \tag{2}$$

The extensional strains are given by $\boldsymbol{v} = \boldsymbol{r}'$.

The equations are closed by specifying linear constitutive relations between the stresses $\boldsymbol{n} = n_1\boldsymbol{d}_1 + n_2\boldsymbol{d}_2 + n_3\boldsymbol{d}_3$, $\boldsymbol{m} = m_1\boldsymbol{d}_1 + m_2\boldsymbol{d}_2 + m_3\boldsymbol{d}_3$, and the strains. For an isotropic rod that is extensible but unshearable these are:

$$m_1 = Bu_1, \quad m_2 = Bu_2, \quad m_3 = Cu_3, \quad n_3 = K(v_3 - 1), \tag{3}$$

where $v_3 = \boldsymbol{v} \cdot \boldsymbol{d}_3$ and B, C and K are the bending, twisting and axial stiffnesses, respectively.

In all this gives a 10-dimensional system of equations for $(n_1, n_2, n_3, m_1, m_2, m_3, q_1, q_2, q_3, q_4)$ with trivial periodic solution (representing the straight but twisted rod)

$$\boldsymbol{p}(s) = \left(0, 0, T, 0, 0, M, 0, 0, \cos\frac{M}{C}s, -\sin\frac{M}{C}s\right), \tag{4}$$

where T and M are the end tension and twisting moment applied axially to the rod, i.e., in the $\boldsymbol{d}_3$ direction.

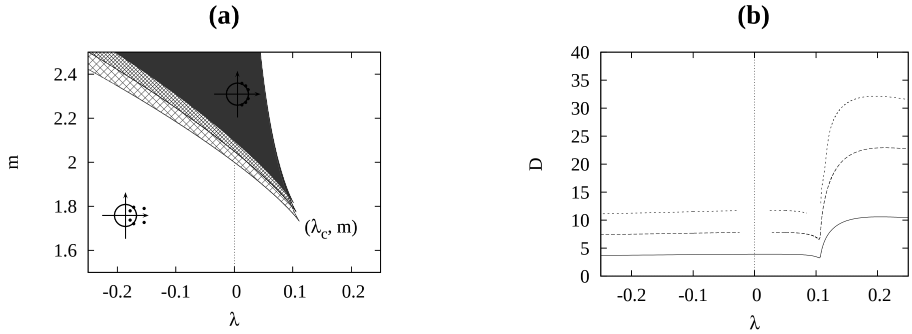

Figure 1: (a) Spectrum of the trivial periodic orbit (4) in the λ-m parameter plane for $\nu := B/C - 1 = 1/3$ and $\gamma := T/K = 0$ (coarse mesh), $\gamma = 0.05$ (fine mesh) and $\gamma = 0.1$ (grey). This elliptic region is bounded by curves of Hamiltonian-Hopf bifurcations terminating in a codimension-two cusp point at (λ_c, m_c). Outside the wedge-shaped region homoclinic orbits may occur. (b) Bifurcation diagram showing curves for primary, 2-pulse, and 3-pulse homoclinic orbits.

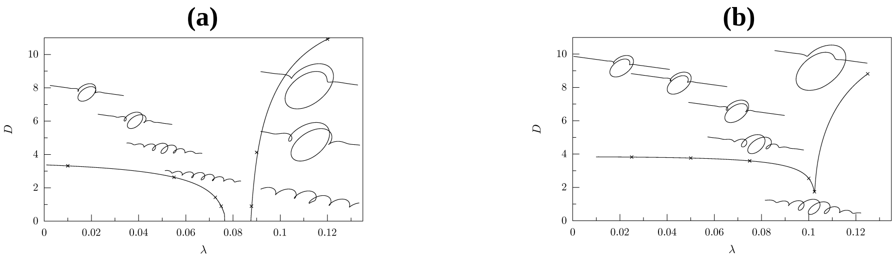

Figure 2: Bifurcation diagrams for $m = 1.90 > m_c$ (a) and $m = 1.81 < m_c$ (b) with rod configurations included. ($\nu = 1/3$, $\gamma = 0.1$.)

Homoclinic bifurcation and instability of the straight rod

The norm condition $\boldsymbol{q} \cdot \boldsymbol{q} = 1$ and the existence of three conserved quantities ($\frac{1}{2}\boldsymbol{n} \cdot \boldsymbol{n} + I\bar{B}\,\boldsymbol{m} \cdot \boldsymbol{e}_3$, $\boldsymbol{n} \cdot \boldsymbol{e}_3$ and $\boldsymbol{m} \cdot \boldsymbol{d}_3$) cause the linearisation about the trivial solution $\boldsymbol{p}$ to separate into a 4D system of trivial dynamics and a 6D system of non-trivial dynamics. The latter has two more Floquet multipliers on the unit circle and the spectrum in the parameter plane of dimensionless $m = M/\sqrt{BT}$ versus $\lambda = I\bar{B}B/(MT)$ is given in Fig. 1(a).

To compute (approximations of) homoclinic orbits we employ the reversing symmetry of the system in a shooting method that chooses initial conditions in the two-dimensional unstable eigenspace of the fixed point corresponding to the trivial periodic solution in an appropriate Poincaré map and shoots into the fixed point set of the reversing symmetry. Special measures are taken to handle the present case of a non-hyperbolic fixed point. 3D localised rod shapes corresponding to the computed homoclinic solutions are shown as insets in Fig. 2.

We find a different multiplicity of homoclinic orbits to that found in previous studies of non-magnetic rods [4]. Due to the persistent rotational symmetry the circle of primary (single-pulse) orbits does not break up under the perturbing effects of magnetic field and extensibility even though the system becomes nonintegrable. As a result of this nonintegrability, circles of additional, higher-order (multi-pulse) homoclinic orbits appear as well. Bifurcation diagrams for the first three, obtained using the continuation code AUTO, are shown in Fig. 1(b), where D is the end-to-end shortening of the rod. Note that for $\lambda = 0$ the system is integrable.

As in previous analyses, critical points of localised buckling are described by Hamiltonian-Hopf bifurcations. However, unlike in previous cases we find that two critical values occur (in either λ or m) inducing localisation-delocalisation behaviour (see Fig. 2). An organising role in the bifurcation behaviour is played by the codimension-two Hamiltonian-Hopf-Hopf bifurcation where these two Hamiltonian-Hopf bifurcations coincide (corresponding to the cusp in Fig. 1(a)). Such a bifurcation requires at least a six-dimensional system of equations and does not seem to have been studied before.

For practical applications the stability implications are of interest. Our results show that the end-loaded straight rod is unstable for parameters inside the wedge-shaped region in Fig. 1(a). Thus the rod becomes unstable at the first bifurcation in Fig. 2(a) for $m > m_c$, but restabilises at larger values of λ, while for $m < m_c$ the rod is stable against localised buckling for any λ (cf. Fig. 2(b)). Coiling behaviour has been reported in certain electrodynamic tether flights.

References

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