CTMS-MAT-13 : Taylor Series

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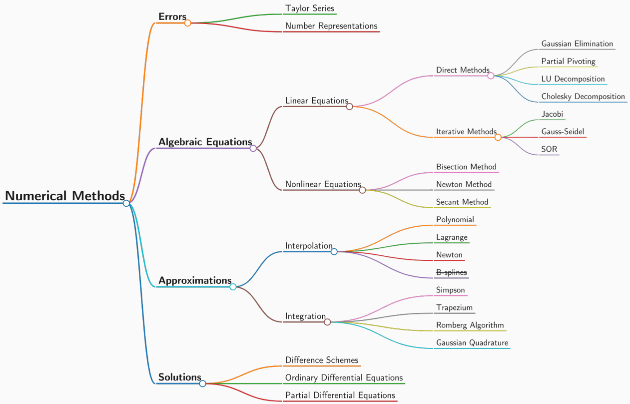
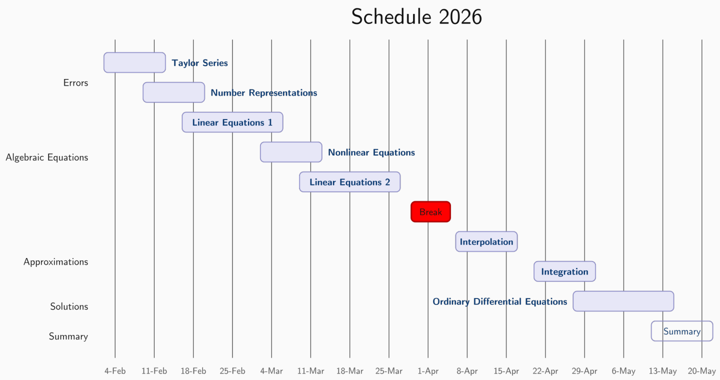
Structure



See homepage
https://djps.github.io/docs/numericalmethods/intro/intro/

Schedule



See notes at

https://djps.github.io/teaching/numerical_methods.pdf

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Taylor Series

Setup

Let $f \in \mathcal{C}^3([a,b])$, i.e. f is three times continuously differentiable on $[a,b]$.

We consider a point $c \in [a,b]$ and the derivatives of f at c:

$$f'(c), \quad f''(c), \quad f^{(3)}(c), \quad \ldots$$

Example

$$f(x) = x^2 \implies f' = 2x, \quad f'' = 2, \quad f^{(3)} = 0.$$

The Taylor Series

The **Taylor series** of f expanded about c is:

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

where $k!$ denotes the factorial:

$$k! = 1 \cdot 2 \cdot 3 \cdots (k-1) \cdot k$$

$$0! = 1, \quad 1! = 1 \qquad\qquad 2! = 2, \quad 3! = 6, \quad \ldots$$

Example: $f(x) = e^x$ at $c = 0$

Since $f^{(k)}(x) = e^x$ for all k, we have:

$$f^{(k)}(0) = e^0 = 1$$

Therefore the Taylor series of e^x is:

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Convergence

Radius of Convergence for e^x

The radius of convergence for e^x is ∞.

This means the Taylor series gives a **very good** approximation of $f(x)$ for *any* x, i.e. the interval $[a, b]$ is $(-\infty, +\infty)$.

What does “very good” mean?

It depends on how many terms k are included in the summation.

When is the Taylor Series Exact?

- The Taylor series **can be exact** for a polynomial (finite sum).
- For some functions it **can serve as the definition** of that function.

Example: $f(x) = 4x^2 + 5x + 7$ at $c = 2$

Computing derivatives at $c = 2$:

$$f(c) = 4 \cdot 4 + 5 \cdot 2 + 7 = 33$$

$$\begin{aligned}
f'(x) &= 8x + 5 \implies f'(c) = 16 + 5 = 21 \\
f''(x) &= 8 \implies f''(c) = 8 \\
f'''(x) &= f'''(c) = 0
\end{aligned}$$

Polynomial Example: Taylor Expansion

Plugging into the formula $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$:

$$= \underbrace{33}_{k=0} + \underbrace{21(x-2)}_{k=1} + \underbrace{\frac{8}{2}(x-2)^2}_{k=2} + 0 + \cdots$$

Expanding:

$$\begin{aligned} &= 33 + 21x - 42 + 4(x^2 - 4x + 4) \\ &= 4x^2 + (21 - 16)x + (16 - 42 + 33) \\ &= 4x^2 + 5x + 7\checkmark \end{aligned}$$

Taylor's Theorem

Theorem

Let $f \in C^{n+1}([a,b])$, i.e. f is $(n+1)$-times differentiable on $[a,b]$. Then for $c, x \in [a,b]$:

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}$$

where ξ is some point depending on x and c (remainder term).

The Remainder Term

The point ξ lies **between** c and x:

$$\xi \in (c, x) \quad \text{for } c < x$$

$$\xi \in (x, c) \quad \text{for } c > x$$

Special case $n = 0$

$$f(x) = f(c) + f'(\xi)(x - c)$$

Setting $c = a, x = b$:

$$\frac{f(b) - f(a)}{b - a} = f'(\xi) \quad \Longrightarrow \quad \textbf{Mean Value Theorem}$$

When Does the Taylor Series Represent f?

We say the Taylor series **represents** f at x if and only if:

1. The Taylor series converges at that point, and
2. The remainder term tends to zero as $n \to \infty$.

That is:

$$\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1} \longrightarrow 0 \quad \text{as } n \to \infty.$$

Convergence of the Taylor Series for e^x

Consider $f(x) = e^x$ at $c = 0$. By Taylor's Theorem:

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^\xi}{(n+1)!} x^{n+1}, \qquad \xi \in (0, x).$$

For $x > 0$, since e^t is increasing and $\xi < x$:

$$e^\xi < e^x =: e^s \quad \text{for some fixed } s > x.$$

Therefore the remainder satisfies:

$$\left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| \le e^s \cdot \frac{s^{n+1}}{(n+1)!}.$$

Remainder $\to 0$ for e^x

We need $\lim_{n\to\infty} e^s \cdot \frac{s^{n+1}}{(n+1)!} = 0.$

Since $(n+1)!$ grows **faster** than any power s^{n+1}:

$$\lim_{n\to\infty} \frac{s^{n+1}}{(n+1)!} = 0$$

$$\implies \quad \lim_{n\to\infty} \left| \frac{e^\xi}{(n+1)!} x^{n+1} \right| = 0.$$

Conclusion

e^x is represented by its Taylor series for all $x \in (-\infty, +\infty)$.

Example: $f(x) = \ln(1+x)$ at $c = 0$

Computing derivatives:

$$\begin{aligned} f(x) &= \ln(1+x) \\ f'(x) &= (1+x)^{-1} \\ f''(x) &= -(1+x)^{-2} \\ f^{(k)}(x) &= (-1)^{k-1}(k-1)!\,(1+x)^{-k} \end{aligned}$$

At $c = 0$: $f(0) = \ln(1) = 0$, so the Taylor series is:

$$f(x) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k + \frac{(-1)^n}{n+1} \cdot \frac{x^{n+1}}{(1+\xi)^{n+1}}$$

Convergence for $\ln(1+x)$ i

The remainder term is:

$$R_n = \frac{(-1)^n}{n+1}\left(\frac{x}{1+\xi}\right)^{n+1}, \qquad \xi \in (0,x).$$

As $n \to \infty$:

$$\lim_{n\to\infty} |R_n| = \lim_{n\to\infty} \frac{1}{(1+n)!}\left|\frac{x}{1+\xi}\right|^{n+1}.$$

If $0 \leq \dfrac{x}{1+\xi} < 1$ the term does **not blow up**, and since $\dfrac{(-1)^n}{(1+n)!} \to 0$:

Convergence for $\ln(1+x)$ ii

$$R_n \to 0 \quad \Longleftrightarrow \quad 0 \le x \le 1.$$

Conclusion

The Taylor series of $\ln(1+x)$ represents the function for $x \in [-1, 1]$.

Numerical Example: Computing $\cos(0.1)$

Taylor series of $\cos(x)$ about $c = 0$:

$$\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

The error (remainder) after n terms satisfies:

$$\left| \cos(x) - \sum_{k=0}^{n} (-1)^k \frac{x^{2k}}{(2k)!} \right| = \left| (-1)^{n+1} \cos(\xi) \, \frac{x^{2(n+1)}}{(2n+2)!} \right|$$

$$\leq \frac{|x|^{2(n+1)}}{(2n+2)!}.$$

$\cos(0.1)$: Taylor Approximations

n	Taylor approx.	Error bound
0	1	$\frac{(0.1)^2}{2} = 0.005$
1	0.995	$\frac{(0.1)^4}{4!} = \frac{0.0001}{24}$
2	0.9950042	$\frac{(0.1)^6}{6!} = \frac{0.000001}{720}$

Note: the error depends on both $|x - c|$ and n.

Computing $\ln 2$ via Taylor Series

Use the Taylor series for $\ln(1+x)$ with $c = 0$, evaluated at $x = 1$:

$$\ln 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \cdots$$

With $k = 8$ terms this gives $\ln 2 \approx 0.63452\ldots$

Problem

This series converges *slowly* — many terms are needed for accuracy.

Instead, use the identity:

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x)$$
$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Choose $x = \frac{1}{3}$ so that $\dfrac{1+x}{1-x} = 2$. Then:

$$\ln 2 = 2\left(\frac{1}{3} + \frac{1}{3^3 \cdot 3} + \frac{1}{3^5 \cdot 5} + \cdots\right).$$

A Faster Series for $\ln 2$ ii

With only $k = 4$ terms:

$$\ln 2 \approx 0.69313.$$

This converges **much faster** than the direct series.